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A REMARK ON FINITE DIMENSIONAL COMPACT CONNECTED MONOIDS

L. W. ANDERSON* AND R. P. HUNTER[1]

ABSTRACT. Let S be a compact n-dimensional monoid. Let A be a compact connected subsemigroup algebraically irreducible from the minimal ideal to the identity of S. Then there exists a closed proper ideal J such that $\dim\{A/A\cap J\}\leqq\dim S-\dim H_1$.

If S is as above, then, according to a result of Wallace [6], the dimension of the maximal subgroup at the identity H_1 cannot exceed $\dim S-1$. In fact, if $\dim H_1=\dim S-1$, then there is a local thread at the identity meeting H_1 at $\{1\}$ [5]. This is a corollary of the following

PROPOSITION 1. *Let S be a compact connected semigroup, with identity 1. Let A be a compact connected subsemigroup algebraically irreducible from $\{1\}$ to the minimal ideal of S. Then there exists a closed proper ideal J such that*

$$\dim\{A/A \cap J\} \leqq \dim S - \dim H_1.$$

In the above, dimension is taken in the sense of Cohen [3]. The canonical reference throughout is [5].

It seems not unreasonable to conjecture that if $\dim A=\dim S-\dim H_1$, notations as above, then some form of uniqueness might be established for A. We include the following example to show that this is, unfortunately, not the case. Indeed we have the following

PROPOSITION 2. *For any $n>0$ there is a compact connected abelian semigroup S of dimension n with identity 1 and zero 0 containing two algebraically irreducible subsemigroups A_1 and A_2 such that $A_1\cap A_2=\{0, 1\}$ such that, given any closed proper ideal J,*

$$\dim A_1/A_1 \cap J = \dim S = \dim A_2/A_2 \cap J.$$

The proof of Proposition 1 will be broken down into a few lemmas some of which may be of interest to the reader.

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* Posthumous

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LEMMA 1. *Let S be a compact monoid with A a compact abelian subsemigroup such that $A \cap H_1 = \{1\}$ and $E(A)$ ordered by $e \leqq f \Rightarrow ef = fe = e$. For $e^2 = e \in A$ let B_e denote the maximal subgroup of A determined by e. Let*

$$Y_e = \{g \mid g \in H_1, ge \in B_e\} = \{g \mid g \in H_1, gB_e = B_e\}.$$

Then Y_e is a closed subgroup of H_1. If H_1 has finite dimension there is a closed proper ideal J such that $e^2 = e \notin A \cap J$ implies dim $Y_e = 0$. *If H_1 is connected, as well as finite dimensional, then J may be chosen so that Y_e is both zero dimensional and central in H_1.*

PROOF. Since the idempotents of A are ordered and A is abelian, the collection of subgroups $\{Y_e\}$ of H_1 is ordered by inclusion. Thus, $e \leqq f$ implies $Y_e \supseteq Y_f$. Clearly we may, without loss of generality, assume that there are idempotents of A, below and arbitrarily close to 1. Since $A \cap H_1 = \{1\}$, it follows from continuity of multiplication that the common part of the groups $\{Y_e\}$, $e \neq 1$, $e \in A$, is precisely $\{1\}$.

Since a compact connected group cannot contain a proper compact connected subgroup of the same finite dimension, it follows that, for some $f^2 = f$, the corresponding Y_f is zero dimensional. Certainly the ideal generated by f can be taken as J.

The last claim of the lemma follows from the following fact: If G is a compact connected finite dimensional group and $\{G_\alpha\}$ is a collection of closed subgroups ordered by inclusion with intersection equal to the identity of G then there is some α such that G_α is zero dimensional and central.

Indeed, G can be given as $L \times C/F$ where L is semisimple, C is connected and abelian and F is central, finite and such that $F \cap C = \{1\}$. If q is the natural homomorphism of $L \times C$ onto G then the collection $\{q^{-1}(G_\alpha)\}$ is ordered by inclusion and $\{\pi(q^{-1}(G_\alpha))\}$ is a nested collection where $\pi: L \times C \to L$ is the projection onto L.

Since L is a compact Lie group, the collection $\pi(q^{-1}(G_\alpha))$ can have only finitely many distinct members. Thus for some α we have that $\pi(q^{-1}(G_\alpha))$ is trivial so $q^{-1}(G_\alpha) \subset C$ so that $G_\alpha \subset q(C) = Z(G)$.

LEMMA 2. *If S is a compact monoid and B a compact subgroup with identity e, then the product H_1B is homeomorphic to a homogeneous space $H_1 \times B/\tilde{Y}$ where $\tilde{Y}$ is a subgroup of $H_1 \times B$ isomorphic with*

$$Y_e(B) = Y_e = \{g \mid g \in H_1, ge \in B\} = \{g \mid g \in H_1, gB = B\}.$$

Thus, if Y_e is central in H_1 and $H_1e \cap B$ lies in the center of B, then H_1B is the underlying space of a compact group.

PROOF. Consider the action of $H_1 \times B$ on Se defined by $((g, b), x) \to gxb^{-1}$. The orbit at e is $H_1eB = H_1B$ and the stability group $\tilde{Y}$ at e is

$$\{(g, b) \mid geb^{-1} = e\} = \{(g, b) \mid ge = b\} = \{(g, (ge)^{-1}) \mid g \in Y_e\}.$$

Now $\tilde{Y}$ and Y_e are isomorphic under the correspondence $(g, b) \leftrightarrow g$. Clearly if g lies in the center of H_1 and $ge = b$ lies in the center of B then (g, b) is in the center of $H_1 \times B$.

LEMMA 3. *Let S be a compact connected monoid of finite dimension. Let H_1 be connected and let B be a compact connected abelian subgroup with identity e such that $Y_e(B) = Y_e$ is zero dimensional and central in H_1. Then H_1B is the underlying space of a compact connected group of dimension* $\dim H_1 + \dim B$. *In particular, if H_1B does not meet the minimal ideal of S then*

$$\dim S > \dim H_1B = \dim H_1 + \dim B.$$

PROOF. The first claim follows from Lemma 2. If we were to have $\dim S = \dim H_1B$ and an element x such that $x(H_1B) \cap H_1B = \square$ we would have a contradiction since we would have $H^{\dim S}(H_1B) = 0$ (see [5, p. 54]). But H_1B is topologically a compact group. With Z as coefficient group we would have $H^{\dim S}(H_1B) \neq 0$.

The following lemma follows from the same argument used in [2].

LEMMA 4. *Let S be a compact monoid, C be the component of $\{1\}$ in H_1, and A be a compact connected subsemigroup algebraically irreducible from $\{1\}$ to the minimal ideal of S. Then* $\text{cl}\langle C, A\rangle$ *the closure of the semigroup generated by C and A, has C for its maximal subgroup at the identity.*

The semigroup $\text{cl}\langle C, A\rangle$ is the compact connected semigroup algebraically irreducible about $C \cup A$.

PROOF OF PROPOSITION 1. From Lemma 4 we may assume H_1 is connected and from Lemma 3 we know that $\dim S > \dim H_1B_e$ so that

$$\dim S \geqq \dim H_1 + \dim B_e + 1,$$

where $e \in E(A) \backslash J$, in accordance with the second part of Lemma 1. However, $\dim(A/A \cap J) = 1 + \sup\{\dim B_e | e \in E(A) \cap J\}$. Thus

$$\dim\{A/A \cap J\} \leqq \dim S = \dim H_1.$$

REMARK. In terms of the above notations one can infer from [4] that $\dim S \geqq \dim H_1B_e + 1$. Then, using only the homogeneous structure of H_1B_e, one concludes that

$$\dim H_1B_e = \dim H_1 + \dim B_e + \dim Y_e = \dim H_1 + \dim B_e,$$

via Lemma 2.

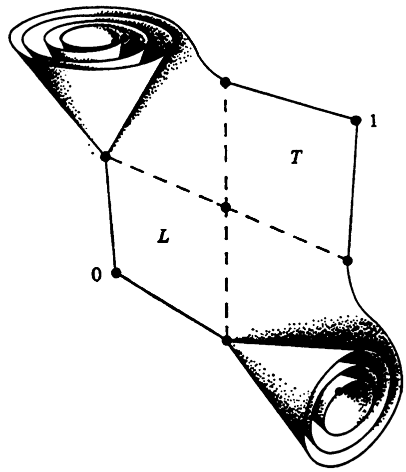

FIGURE 1

PROOF OF PROPOSITION 2. Let S be composed of a half-open interval winding upon a circle group K so that $S/K=S/\mathscr{H}=[0, e, u]$ where $[0, e]$ and $[e, u]$ are ordinary unit intervals.

Now form $S\times S$ as usual, and decompose $S\times S$ according to $\mathscr{C}$, where the classes of $\mathscr{C}$ are sets of the form $K\times\{t\}$, $\{t\}\in K$ or $K\times K$ where $t\in eSe$, and points otherwise. (See Figure 1.)

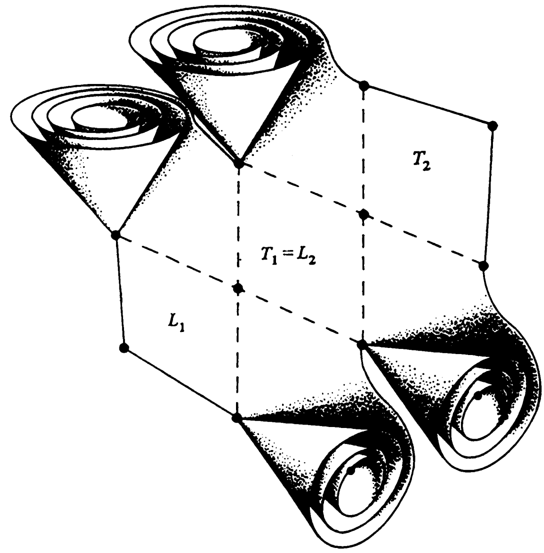

FIGURE 2

This is a modified cone construction similar to those in [**1**]. Note $S_1 = S \times S/\mathscr{C}$ contains two copies of S meeting at the identity, and two isomorphic algebraically irreducible semigroups from zero to identity meeting at just zero and identity. Both have dimension two.

Now in terms of the figure, the ideal $L = L_1$ is isomorphic to the submonoid $T = T_1$. Both of these are isomorphic to the cartesian product of two unit intervals. Thus if S_2 is another copy of S_1 there is an isomorphism ϕ between L_2 and T_1, and S_2 may be attached to S_1 via ϕ. (See Figure 2.)

The rest of the construction is now standard. The semigroups S_n are taken converging to a point taken as identity. Clearly in place of the circle group one may use any compact connected finite dimensional abelian group.

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Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802