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ON THE RADIAL AND NONTANGENTIAL MAXIMAL FUNCTIONS FOR THE DISC

RICHARD L. WHEEDEN[1]

ABSTRACT. Positive powers of the radial and nontangential maximal functions of a function which is harmonic or analytic in the unit disc are shown to have equivalent integrals with respect to Borel measures satisfying the growth condition $\mu(2I)\leqq c\mu(I)$ for every interval I.

Let $u(z)$ be a function which is harmonic or analytic in the unit disc $D=\{z:z=re^{ix},\ 0\leqq r<1,\ -\pi<x\leqq\pi\}$. For $0<\alpha<1$, let $\Gamma_\alpha(x)$ denote the open subset of D bounded by the two tangent lines from e^{ix} to the circle $|z|=\alpha$ and the longer of the two arcs of $|z|=\alpha$ between the points of tangency. Let

$$N_0(u)(x) = \sup_{0\leqq r<1} |u(re^{ix})| \quad \text{and} \quad N_\alpha(u)(x) = \sup_{z\in\Gamma_\alpha(x)} |u(z)|, \qquad 0 < \alpha < 1,$$

denote the radial and nontangential maximal functions of u.

Let μ be a nonnegative, periodic, finite Borel measure on $|z|=1$. If I is an interval (arc) on $|z|=1$ and $a>0$, let aI denote the interval concentric with I whose length is a times that of I. We assume that μ satisfies

$$\mu(2I) \leqq c\mu(I) \tag{1}$$

for every interval I, where c is a positive constant independent of I.

A related condition, introduced in an equivalent form by C. Fefferman, is that there exist positive constants c and ε such that for every interval I and every Lebesgue measurable subset E of I,

$$\mu(E)/\mu(I) \leqq c(|E|/|I|)^\varepsilon, \tag{2}$$

where $|E|$ denotes the Lebesgue measure of E. Such a measure is absolutely continuous with respect to Lebesgue measure, and a known argument (see Remark (d), §1, of [5]) shows it satisfies (1). On the other hand, by a result of C. Fefferman and B. Muckenhoupt [3], there are measures which satisfy (1) but not (2). Our result here is that powers of $N_0(u)$ and $N_\alpha(u)$ have equivalent integrals with respect to measures μ satisfying (1).

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THEOREM. *If $u(z)$ is harmonic or analytic in D, μ satisfies* (1), $0<p<\infty$ *and* $0<\alpha<1$ *then*

$$\int_{-\pi}^{\pi}\{N_\alpha(u)(x)\}^p\,d\mu(x) \leqq c\int_{-\pi}^{\pi}\{N_0(u)(x)\}^p\,d\mu(x),$$

with c independent of u.

In case $d\mu(x)=dx$, this result is contained in Corollary 2, p. 170 of **[4]**. The proof given there can be easily adapted to the general case for measures satisfying (2). For such μ, the finiteness of the integral on the left in the Theorem is equivalent to the statement that u belongs to a weighted version of the classical Hardy space H^p (see §1 of **[5]**). To prove the Theorem for a measure which only satisfies (1), we will need a different argument, based on observations of D. Burkholder and R. Gundy **[1]**.

For $z \in D$ and $0<\delta<1-|z|$, $B(z, \delta)$ will denote the subset $\{\zeta : |z-\zeta|<\delta\}$ of D. Different positive constants will be denoted by the same c followed by the parameters on which they depend.

LEMMA 1. *Let $u(z)$ be a continuous function defined for $z \in D$ and let* $0<\alpha<\beta<1$. *If μ satisfies* (1) *there is a constant $c_1=c_1(\alpha, \beta, \mu)$ so that for all $y>0$*

$$\mu\{x : N_\beta(u)(x) > y\} \leqq c_1\mu\{x : N_\alpha(u)(x) > y\}.$$

For other forms of this lemma, see Lemma 2 of **[2]** and Lemma 1 of **[5]**.

PROOF. Choose a positive number $t=t(\alpha, \beta)$ so small that for any interval I on $|z|=1$ with $|I|\leqq t$,

$$D - \bigcup_{x\notin I} \Gamma_\alpha(x) \subset \{(\beta+1)/2 < |z| < 1\}.$$

Fix $y>0$ and write the open set $\{N_\alpha(u)>y\}$ as the union of nonoverlapping intervals I_j. If $|I_j|>t$ for any j, then by (1) there is a constant $c_1=c_1(t, \mu)$ such that $\mu(I_j)\geqq c_1\mu(-\pi, \pi)$. Therefore, $\mu\{N_\alpha(u)>y\}\geqq c_1\mu\{N_\beta(u)>y\}$, and Lemma 1 is proved. If all I_j have $|I_j|\leqq t$, then since

$$\{z : |u(z)| > y\} \subset \bigcup_j \left(D - \bigcup_{x\notin I_j} \Gamma_\alpha(x)\right),$$

a simple geometric argument shows that

$$\{N_\beta(u) > y\} \subset \bigcup_j aI_j,$$

where a is a constant larger than 1 which depends on α and β but not on j.

Hence

$$\mu\{N_\beta(u) > y\} \leqq \sum_j \mu(aI_j) \leqq c_1 \sum_j \mu(I_j) \quad \text{(by (1))}$$
$$= c_1\mu\{N_\alpha(u) > y\}.$$

LEMMA 2. *Let $u(z)$ be a bounded function which is harmonic or analytic in D and let $0<\alpha<\beta<1$ and $k>1$. If μ satisfies* (1) *there is a constant $c_2=c_2(\alpha, \beta, k, \mu)$ such that for all $y>0$*

$$\mu\{x:N_\alpha(u)(x) > y,\ N_\beta(u)(x) < kN_\alpha(u)(x)\} \leqq c_2\mu\{x:N_0(u)(x) > y/4\}.$$

This lemma is an adaptation of an argument given in [**1**].

PROOF. With each x associate a point $z \in \Gamma_\alpha(x)$ such that $|u(z)|> \frac{1}{2}N_\alpha(u)(x)$. Fix $y>0$ and let $S=\{N_\alpha(u)>y,\ N_\beta(u)<kN_\alpha(u)\}$. We first claim that there is a positive constant $\varepsilon_0=\varepsilon_0(\alpha, \beta, k)$ such that $|u(\zeta)|> \frac{1}{4}N_\alpha(u)(x)$ for all $x \in S$ and $\zeta \in B(z, \varepsilon_0\delta)$, $\delta=1-|z|$. To see this, fix $x \in S$ and let $a=|u(z)|$, so that $N_\beta(u)(x)<2ka$. Since $z \in \Gamma_\alpha(x)$ and $\beta>\alpha$, there is by the geometry of the situation a positive number $s=s(\alpha, \beta)$, independent of x and $z \in \Gamma_\alpha(x)$, such that $B(z, s\delta)$ lies in $\Gamma_\beta(x)$. In particular, $|u(\zeta)|<2ka$ if $\zeta \in B(z, s\delta)$. If $\zeta \in B(z, s\delta/2)$ then $B(\zeta, s\delta/2)\subset B(z, s\delta)$, and therefore, since by [7, p. 275], there is an absolute constant c so that

$$|\nabla u(\zeta)| \leqq \frac{c}{s\delta}\left(|B(\zeta, s\delta/2)|^{-1} \iint\limits_{B(\zeta,s\delta/2)} |u(\tau)|^2\, d\tau\right)^{1/2},$$

we have

$$|\nabla u(\zeta)| \leqq c(2ka)/s\delta, \qquad \zeta \in B(z, s\delta/2).$$

Choose ε_0 satisfying $0<\varepsilon_0<\mathrm{Min}[s/2, s/8ck]$. If $\zeta \in B(z, \varepsilon_0\delta)$, we obtain from the last inequality and the mean-value theorem applied to u between z and ζ (or applied separately to the real and imaginary parts of u if u is analytic) that

$$|u(\zeta)| \geqq a - 2(\varepsilon_0\delta)(c2ka/s\delta) = a(1 - \varepsilon_0 4ck/s)$$
$$> a/2 > N_\alpha(u)(x)/4.$$

This proves our claim and, since $x \in S$, that $|u(\zeta)|>y/4$ for $\zeta \in B(z, \varepsilon_0\delta)$. For $x \in S$, let $J(x)$ denote the interval which is the projection of $B(z, \varepsilon_0\delta)$ onto $|z|=1$, and let $I(x)$ denote the smallest interval with center x which contains $J(x)$. By the geometry of the situation,

$$|J(x)| \geqq c\,|I(x)|, \qquad c = c(\alpha, \beta, k) > 0,$$

and therefore by (1)

$$\mu(J(x)) \geqq c\mu(I(x)), \qquad c = c(\alpha, \beta, k, \mu) > 0. \tag{3}$$

Moreover, by our claim,

$$J(x) \subset \{x : N_0(u)(x) > y/4\}. \tag{4}$$

The intervals $I(x)$ cover S, so by p. 304 of [6] we may select a positive integer m and points $x_i \in S$, $i=1, 2, \cdots$, in such a way that $S \subset \bigcup_i I(x_i)$, and no point is contained in more than m different $I(x_i)$'s. Then

$$\begin{aligned} \mu(S) &\leqq \sum_i \mu(I(x_i)) \leqq c \sum_i \mu(J(x_i)), \qquad c = c(\alpha, \beta, k, \mu) \quad \text{(by (3))} \\ &\leqq mc\mu\left(\bigcup_i J(x_i)\right) \leqq mc\mu\{x : N_0(u)(x) > y/4\} \qquad \text{(by (4))}. \end{aligned}$$

This completes the proof of Lemma 2.

To prove the Theorem, first suppose that $u(z)$ is bounded and harmonic or analytic in D. For $0<\alpha<\beta<1$ and $k>1$,

$$\begin{aligned} &\mu\{N_\alpha(u) > y\} \\ &\quad \leqq \mu\{N_\alpha(u) > y, N_\beta(u) < kN_\alpha(u)\} + \mu\{N_\alpha(u) > y, N_\beta(u) \geqq kN_\alpha(u)\}. \end{aligned}$$

Therefore, by Lemma 2, there exists $c_2=c_2(\alpha, \beta, k, \mu)$ such that

$$\begin{aligned} &p\int_0^\infty y^{p-1}\mu\{N_\alpha(u) > y\}\, dy \\ &\qquad \leqq c_2 p\int_0^\infty y^{p-1}\mu\{N_0(u) > y/4\}\, dy \\ &\qquad\quad + p\int_0^\infty y^{p-1}\mu\{N_\alpha(u) > y, N_\beta(u) \geqq kN_\alpha(u)\}\, dy. \end{aligned} \tag{5}$$

The last integral equals $\int_{\{N_\beta(u) \geqq kN_\alpha(u)\}} N_\alpha(u)^p\, d\mu$, which is at most

$$k^{-p}\int_{-\pi}^{\pi} N_\beta(u)^p\, d\mu \leqq c_1^p k^{-p}\int_{-\pi}^{\pi} N_\alpha(u)^p\, d\mu, \qquad c_1 = c_1(\alpha, \beta, \mu),$$

by Lemma 1. Since u is bounded all these integrals are finite. From (5),

$$\int_{-\pi}^{\pi} N_\alpha(u)^p\, d\mu \leqq 4^p c_2\int_{-\pi}^{\pi} N_0(u)^p\, d\mu + c_1^p k^{-p}\int_{-\pi}^{\pi} N_\alpha(u)^p\, d\mu.$$

The first and last integrals here are the same. Since c_1 is independent of k, we may choose k so large that $c_1^p k^{-p} \leqq \frac{1}{2}$, and the Theorem follows in this case.

The case of arbitrary harmonic or analytic u can be deduced from the case of bounded u by putting $u_r(z)=u(rz)$, $0<r<1$. Since u_r is bounded and $N_0(u_r)(x) \leqq N_0(u)(x)$, we have

$$\int_{-\pi}^{\pi} N_\alpha(u_r)^p \, d\mu \leqq c \int_{-\pi}^{\pi} N_0(u_r)^p \, d\mu \leqq c \int_{-\pi}^{\pi} N_0(u)^p \, d\mu,$$

with c independent or r and u. The result follows from the monotone convergence theorem by letting $r \to 1$.

References

1. D. Burkholder and R. Gundy, *Boundary behavior of harmonic functions in a half-space and Brownian motion*, Ann. Inst. Fourier. **23** (1973).

2. ———, *Distribution function inequalities for the area integral*, Studia Math. **44** (1972), 527–544.

3. C. Fefferman and B. Muckenhoupt, *Two nonequivalent conditions for weight functions*, Proc. Amer. Math. Soc. (to appear).

4. C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

5. R. Gundy and R. Wheeden, *Weighted integral inequalities for the nontangential maximal function, Lusin area integral and Walsh-Paley series*, Studia Math. (to appear).

6. M. de Guzman, *A covering lemma with applications to differentiability of measures and singular integral operators*, Studia Math. **34** (1970), 299–317. MR **41** #8621.

7. E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

Department of Mathematics, University Heights Campus, Rutgers University, New Brunswick, New Jersey 08903