

THE k TH CONJUGATE POINT FUNCTION FOR AN EVEN ORDER LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. For an even order, two term equation $L_n y = p(x)y$, $p(x) > 0$, x in $[0, \infty)$, the k th conjugate point function $\eta_k(a)$ is defined and is shown to be a strictly increasing continuous function with domain $[0, b)$ or $[0, \infty)$. Extremal solutions are defined as nontrivial solutions with $n-1+k$ zeros on $[a, \eta_k(a)]$, and are shown to have exactly $n-1+k$ zeros, with even order zeros at a and $\eta_k(a)$ and exactly $k-1$ odd order zeros in $(a, \eta_k(a))$, thus establishing that $\eta_k(a) < \eta_{k+1}(a)$.

The differential equation considered in this paper is defined as follows: Let p_1, \dots, p_{n+1} be positive continuous functions defined on $[0, \infty)$, and let A_0 denote the set of all continuous functions defined on $[0, \infty)$. For y in A_0 , define

$$(1.1) \quad L_0 y = p_1 y.$$

Assume that A_i and $L_i y$ have been defined for $i \leq k-1$, and let A_k denote the set of all functions y for which $L_{k-1} y$ has a continuous derivative on $[0, \infty)$. For y in A_k , define

$$(1.2) \quad L_k y = p_{k+1} (L_{k-1} y)'$$

The differential equation with which we are concerned is

$$(1.3) \quad L_n y = p y,$$

where p is a positive continuous function defined on $[0, \infty)$, and $n \geq 4$ is an even integer (cf. [3], [5], [6]).

From (1.2) and (1.3) it follows that L_n has the factored form

$$(1.4) \quad L_n y = p_{n+1} (p_n (\dots p_2 (p_1 y)' \dots))'$$

For y in A_n , the function $L_i y$ is said to have a zero of multiplicity k at $x=a$ if

$$L_i y(a) = \dots = L_{k-1+i} y(a) = 0 \quad \text{and} \quad L_{k+i} y(a) \neq 0.$$

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Since the functions p_1, \dots, p_{n+1} are positive, it follows that if $L_{k-1}y(a) = L_{k-1}y(b) = 0$ for some $a < b$, then there is a number c in (a, b) for which $L_k y(c) = 0$. Moreover, if $L_i y$ has a zero of multiplicity k at $x = a$, then $L_i y$ changes sign at $x = a$ if and only if k is odd.

Denote by $N(k, a)$ the set of all points $x > a$ for which there is a nontrivial solution of (1.3), with zeros at a and x , having at least $n - 1 + k$ zeros in $[a, x]$. The k th conjugate point of a , denoted by $\eta_k(a)$, is defined to be the infimum of the set $N(k, a)$; if $N(k, a)$ is empty, then $\eta_k(a) = +\infty$. Leighton and Nehari [4] studied the functions η_k extensively for the equation $(ry'')'' - py = 0$ which is a special case of (1.3) with $n = 4$, $p_3 = r$, and $p_i = 1$ if $i \neq 3$.

In this paper we establish the existence of $\eta_k(a)$ whenever $N(k, a)$ is not empty, and we consider the properties of η_k as a function of a . We also investigate the properties of the solutions which are extremal for $\eta_k(a)$, in the following sense. A solution y of (1.3) will be called extremal if y has a zero at a , a zero at $\eta_k(a)$ and $n - 1 + k$ zeros in $[a, \eta_k(a)]$.

We can now state the main results.

THEOREM 1. *If y is an extremal solution for $\eta_k(a)$, then y has even order zeros at a and at $\eta_k(a)$; y has exactly $n - 1 + k$ zeros in $[a, \eta_k(a)]$, with exactly $k - 1$ odd order zeros in $(a, \eta_k(a))$; y is never zero in $(0, a)$ or in $(\eta_k(a), \infty)$.*

THEOREM 2. *As a function of a , η_k is a strictly increasing continuous function whose domain is of the form $[0, b)$, or $[0, \infty)$.*

In order to establish Theorem 1, we will require the following results. For notational purposes, if y is a solution with $n - 1 + k$ zeros at r points $x_1 < \dots < x_r$ we will denote by $m(x_i)$ the multiplicity of the zero of y at x_i . Define the number

$$(1.5) \quad M(y) = \sum_{i \in I} m(x_i) + \sum_{i \in J} [m(x_i) - 1]$$

where

$$I = \{i: m(x_i) \text{ is even}\} \quad \text{and} \quad J = \{i: m(x_i) \text{ is odd}\}.$$

LEMMA 1. *If $N(m, a)$ is nonempty, then for each $k = 1, \dots, m$, there exists a k th conjugate point $\eta_k(a)$ and a nontrivial solution y_k having the following properties.*

- (i) $a < \eta_k(a) \leq \eta_{k+1}(a)$, for $k = 1, \dots, m - 1$.
- (ii) y_k has at least $n - 1 + k$ zeros in $[a, \eta_k(a)]$.

(iii) *No nontrivial solution of (1.3) having a zero at a has more than $n-2+k$ zeros in $[a, \eta_k(a)]$.*

For the proof, we observe that if $N(m, a)$ is nonempty then $N(k, a)$ is nonempty for $k \leq m$, and if $x_0 \in N(m, a)$ then there exist $x_1 \leq x_2 \leq \dots \leq x_k \leq x_0$ such that $x_i \in N(i, a)$. Thus $\eta_k(a)$ exists for each $k < m$ and $\eta_k(a) \leq \eta_{k+1}(a)$, establishing part (i). Part (iii) is a direct consequence of the definition of $\eta_k(a)$. If $N(k, a)$ is finite, or if there exists $\varepsilon > 0$ such that the intersection of $N(k, a)$ with $(\eta_k(a), \eta_k(a) + \varepsilon)$ is empty, then (ii) is immediate. Otherwise we observe that there exists a sequence $\{x_i\}$ converging monotonically to $\eta_k(a)$ and a sequence of solutions $\{y_i\}$ having zeros at a and x_i with at least $n-1+k$ zeros in $[a, x_i]$. With no loss of generality, we may assume that y_i has zeros at $a = t_{i_1} < \dots < t_{i_r} = x_i$ and that there are integers m_1, \dots, m_r with $m_1 + \dots + m_r \geq n-1+k$ such that $m(t_{i_j}) = m_j$ for all $j = 1, \dots, r$, and all i . Normalizing each solution y_i , we may apply standard compactness arguments to obtain a nontrivial solution y of (1.3) and a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ such that $L_q y_{i_j}$ converges uniformly to $L_q y$, for each $q = 0, \dots, n-1$, on $[a, x_1]$. Since limit points of the zeros $L_q y_{i_j}$ are zeros of $L_q y$ we have that y must have at least $n-1+k$ zeros in $[a, \eta_k(a)]$ with a zero at a and a zero at $\eta_k(a)$.

The following lemma, stated without proof, is due to Levin [5].

LEMMA 2. *There does not exist a nontrivial solution of (1.3) satisfying the following boundary conditions at $x_1 < \dots < x_r$.*

$$\begin{aligned} L_i y(x_j) &= 0, \quad i = 0, \dots, m(x_j) - 1, \quad j = 1, \dots, r \\ &\text{if } m(x_i) \text{ and } m(x_r) \text{ are odd and} \\ &m(x_i) \text{ is even for } i = 2, \dots, r-1. \end{aligned}$$

It is an immediate consequence of Lemma 2 that if y is a nontrivial solution of (1.3) with $n-1+k$ zeros at the points $x_1 < \dots < x_r$ then $M(y) \leq n$, the number of odd order zeros of y must exceed $k-2$ and will equal $k-1$ only if $m(x_1)$ and $m(x_r)$ are even.

LEMMA 3. *If y is a nontrivial solution of (1.3) having $n-1+k$ zeros at $x_1 < \dots < x_r$ such that $M(y) < n$ then there exists a nontrivial solution of (1.3) with $n-1+k$ zeros on $[x_1, x_r]$.*

There are four cases to the proof, depending on whether the zeros at x_1 and x_r are of even or odd multiplicity. Each case is treated in a similar fashion, so we will demonstrate the case in which $m(x_1)$ is even and $m(x_r)$ is odd. Let y_1, \dots, y_n be a fundamental set of solutions of (1.3). By Lemma

2, the matrix

$$Y(\varepsilon) = \begin{bmatrix} L_0 y_1(x_1) & L_0 y_2(x_1) & \cdots & L_0 y_n(x_1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_{q_1} y_1(x_1) & L_{q_1} y_2(x_1) & \cdots & L_{q_1} y_n(x_1) \\ L_0 y_1(x_2) & L_0 y_2(x_2) & \cdots & L_0 y_n(x_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_0 y_1(x_r - \varepsilon) & L_0 y_2(x_r - \varepsilon) & \cdots & L_0 y_n(x_r - \varepsilon) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_{q_r} y_1(x_r - \varepsilon) & L_{q_r} y_2(x_r - \varepsilon) & \cdots & L_{q_r} y_n(x_r - \varepsilon) \end{bmatrix}$$

is nonsingular for all $\varepsilon > 0$ sufficiently small, where $q_1 = m(x_1)$; and for $i = 2, \dots, r-1$, $q_i = m(x_i) - 1$ if $m(x_i)$ is even, $q_i = m(x_i) - 2$ if $m(x_i)$ is odd and $m(x_i) \geq 3$, and $q_r = n - M(y) - 2$. Then for each $\varepsilon > 0$ there is a nontrivial solution vector $c(\varepsilon) = \text{col}(c_1(\varepsilon), \dots, c_n(\varepsilon))$ of $Y(\varepsilon)c(\varepsilon) = Y(0)c$ where $c = \text{col}(c_1, \dots, c_n)$ and $y = \sum_{i=1}^n c_i y_i$ is the solution with $n-1+k$ zeros in $[x_1, x_r]$. Letting $y(x, \varepsilon) = \sum_{i=1}^n c_i(\varepsilon) y_i(x)$, we have that as $\varepsilon \rightarrow 0$, $L_j y(x, \varepsilon)$ converges uniformly to $L_j y^*(x)$ for $j = 0, \dots, n-1$ where y^* is a nontrivial solution of (1.3). Since $y - y^*$ satisfies the boundary conditions of Lemma 2, it must be the case that $y(x) = y^*(x)$ for all x . Now $y(x, \varepsilon)$ has an even order zero at each of the points x_2, \dots, x_{r-1} , and hence for ε sufficiently small, $y(x, \varepsilon)$ must change sign near each odd order zero of y . A simple count establishes that $y(x, \varepsilon)$ must have $n-1+k$ zeros in $[x_1, x_r - \varepsilon]$.

COROLLARY 3.1. *If y is an extremal solution of (1.3) for $\eta_k(a)$, then the zeros at a and $\eta_k(a)$ are of even multiplicity.*

If either a or $\eta_k(a)$ is of odd multiplicity, then $M(y) < n$, which contradicts the fact that y is extremal.

COROLLARY 3.2. *If y is an extremal solution for $\eta_k(a)$, then y has exactly $n-1+k$ zeros in $[a, \eta_k(a)]$.*

If y has $n-1+r$ zeros in $[a, \eta_k(a)]$, and if $r > k$, then $m(\eta_k(a)) > r-k$. Using the techniques of Lemma 3 with the multiplicity $m(\eta_k(a)) - 1$ at $\eta_k(a)$ yields a solution with $n-2+r$ zeros in $[a, \eta_k(a) - \varepsilon]$.

If an extremal solution for $\eta_k(a)$ has m odd order zeros, then clearly $M(y) + m = n - 1 + k$, so that $m = k - 1$. If y has a zero in either $(0, a)$ or $(\eta_k(a), \infty)$, then y satisfies the boundary conditions of Lemma 2, yielding a contradiction. This completes the proof of Theorem 1.

LEMMA 4. *If $\eta_k(b) < \infty$, then there exists $\delta > 0$ such that for a in $(b - \delta, b + \delta)$, $\eta_k(a) < \infty$.*

Let y be an extremal solution for $\eta_k(b)$ with zeros at $b = x_1 < \dots < x_r = \eta_k(b)$. Then, for each ε , sufficiently small, we define $y(x, \varepsilon)$ to be a solution of (1.3) having zeros of multiplicity $m(x_i) - 1$ at x_i if $i = 1, r$ or if $m(x_i)$ is odd, a zero of multiplicity $m(x_i)$ at x_i if $m(x_i)$ is even, $1 < i < r$, and zero at $b + \varepsilon$. Then we may write

$$y(x, \varepsilon) = \sum_{i=1}^n c_i(\varepsilon) y_i(x)$$

where $\{y_1, \dots, y_n\}$ is a fundamental set of solutions of (1.3), and with no loss of generality,

$$\sum_{i=1}^n c_i(\varepsilon)^2 = 1.$$

Then as $\varepsilon \rightarrow 0$, $L_j y(x, \varepsilon)$ converges uniformly to $L_j y^*(x)$, where y^* is a nontrivial solution of (1.3) satisfying

$$y^*(x) = \sum_{i=1}^n c_i y_i(x), \quad \sum_{i=1}^n c_i^2 = 1.$$

If $y^*(x) \neq ky(x)$ for all x , then there is a nontrivial linear combination of y^* and y satisfying the boundary conditions in Lemma 2, which is a contradiction. Then there is a $\delta > 0$ such that if $|\varepsilon| < \delta$, $y(x, \varepsilon)$ changes sign near each odd order zero of y^* and near $\eta_k(b)$ since $y(x, \varepsilon)$ has an odd order zero at $\eta_k(b)$ and y^* has an even order zero there. A simple count yields that $y(x, \varepsilon)$ has $n - 1 + k$ zeros in $[b + \varepsilon, \eta_k(b) + \delta]$, and this completes the proof, since a similar argument holds for $b - \varepsilon$.

If we define $y(x, \varepsilon)$ as in Lemma 4 at the points x_1, \dots, x_r and require that $y(\eta_k(b) - \varepsilon, \varepsilon) = 0$ rather than $y(b - \varepsilon) = 0$, for $\varepsilon > 0$ then we obtain the following.

LEMMA 5. *If $\eta_k(b)$ exists, then there is a sequence $\{a_i\}$ converging to b in $(b - \delta, b)$ such that $\eta_k(a_i) < \eta_k(b)$ for each i .*

From the preceding discussion, we have for each $\varepsilon > 0$, sufficiently small, that there exists a nontrivial solution $y(x, \varepsilon)$ of (1.3) with $n - 1 + k$ zeros in $[b - \varepsilon, \eta_k(b)]$. From Theorem 1, $y(x, \varepsilon)$ is not an extremal solution for $\eta_k(b - \varepsilon)$, hence $\eta_k(b - \varepsilon) < \eta_k(b)$.

COROLLARY 5.1. *If $\eta_k(x) < \infty$ for all $x \in [a, b]$ then $\eta_k(a) < \eta_k(b)$.*

Suppose to the contrary that $\eta_k(a) \geq \eta_k(b)$. For some $\varepsilon > 0$, $\eta_k(b - \varepsilon) < \eta_k(b)$, $b - \varepsilon > a$ and $S = \{x > a : \eta_k(x) < \eta_k(b - \varepsilon)\}$ is nonempty, and hence if $d = \inf S$, then $\eta_k(d) = \eta_k(b - \varepsilon)$. If $a < d$, then $\eta_k(x) \geq \eta_k(b - \varepsilon) \geq \eta_k(d)$ for all x in (a, d) , contradicting Lemma 6. Thus $a = d$. Let x_i be a sequence in S converging monotonically to a . Then arguments of Lemma 1 yield a sequence of extremal solutions y_i converging to a solution y of (1.3) having $n - 1 + k$ zeros in $[a, \eta_k(b - \varepsilon)]$. But this contradicts the definition of $\eta_k(a)$ since $\eta_k(a) \geq \eta_k(b) > \eta_k(b - \varepsilon)$.

COROLLARY 5.2. *If $\eta_k(b) < \infty$, then $\eta_k(x) < \infty$ for all $x \leq b$.*

If, to the contrary, there exists an $x < b$ for which $N(x, k)$ is empty, let $a = \sup\{x < b : N(x, k) \text{ is empty}\}$. Then $a < b$, and $\eta_k(x) < \infty$ for all x in (a, b) . From Corollary 6.1, there is a sequence $\{a_i\}$ in (a, b) such that $a_{i+1} < a_i$, $\eta_k(a_{i+1}) < \eta_k(a_i)$ and a_i converges to a as i tends to ∞ . Then the techniques of Lemma 1 yield a solution y of (1.3) such that $y(a) = 0$ and y has $n - 1 + k$ zeros on $[a, \eta_k(b)]$, contradicting Lemma 4.

COROLLARY 5.3. *η_k is continuous.*

If as $x \rightarrow a - 0$, $\eta_k(x) \rightarrow L < \eta_k(a)$, then the arguments of Lemma 1 yield a nontrivial solution of (1.3) with $n - 1 + k$ zeros in $[a, \eta_k(a)]$ since η_k is increasing. If as $x \rightarrow a + 0$, $\eta_k(x) \rightarrow L > \eta_k(a)$, let $\delta = \frac{1}{2}(L - \eta_k(a))$. By Lemma 4, there exists $\varepsilon > 0$ and a solution of $y(x, \varepsilon)$ having $n - 1 + k$ zeros in $[a + \varepsilon, \eta_k(a) + \delta]$. This contradicts the fact that η_k is increasing.

Corollaries 5.1, 5.2, and 5.3 complete the proof of Theorem 2.

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