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THE kTH CONJUGATE POINT FUNCTION FOR AN EVEN ORDER LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. For an even order, two term equation $L_n y = p(x)y$, $p(x) > 0$, x in $[0, \infty)$, the kth conjugate point function $\eta_k(a)$ is defined and is shown to be a strictly increasing continuous function with domain $[0, b)$ or $[0, \infty)$. Extremal solutions are defined as nontrivial solutions with $n-1+k$ zeros on $[a, \eta_k(a)]$, and are shown to have exactly $n-1+k$ zeros, with even order zeros at a and $\eta_k(a)$ and exactly $k-1$ odd order zeros in $(a, \eta_k(a))$, thus establishing that $\eta_k(a) < \eta_{k+1}(a)$.

The differential equation considered in this paper is defined as follows: Let $p_1, \cdots, p_{n+1}$ be positive continuous functions defined on $[0, \infty)$, and let A_0 denote the set of all continuous functions defined on $[0, \infty)$. For y in A_0, define

$$L_0 y = p_1 y. \tag{1.1}$$

Assume that A_i and $L_i y$ have been defined for $i \leqq k-1$, and let A_k denote the set of all functions y for which $L_{k-1}y$ has a continuous derivative on $[0, \infty)$. For y in A_k, define

$$L_k y = p_{k+1}(L_{k-1}y)'. \tag{1.2}$$

The differential equation with which we are concerned is

$$L_n y = py, \tag{1.3}$$

where p is a positive continuous function defined on $[0, \infty)$, and $n \geqq 4$ is an even integer (cf. [3], [5], [6]).

From (1.2) and (1.3) it follows that L_n has the factored form

$$L_n y = p_{n+1}(p_n(\cdots p_2(p_1 y)' \cdots)')'. \tag{1.4}$$

For y in A_n, the function $L_i y$ is said to have a zero of multiplicity k at $x = a$ if

$$L_i y(a) = \cdots = L_{k-1+i}y(a) = 0 \quad \text{and} \quad L_{k+i}y(a) \neq 0.$$

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Since the functions $p_1, \cdots, p_{n+1}$ are positive, it follows that if $L_{k-1}y(a)=L_{k-1}y(b)=0$ for some $a<b$, then there is a number c in (a, b) for which $L_k y(c)=0$. Moreover, if $L_i y$ has a zero of multiplicity k at $x=a$, then $L_i y$ changes sign at $x=a$ if and only if k is odd.

Denote by $N(k, a)$ the set of all points $x>a$ for which there is a nontrivial solution of (1.3), with zeros at a and x, having at least $n-1+k$ zeros in $[a, x]$. The kth conjugate point of a, denoted by $\eta_k(a)$, is defined to be the infimum of the set $N(k, a)$; if $N(k, a)$ is empty, then $\eta_k(a)=+\infty$. Leighton and Nehari [4] studied the functions η_k extensively for the equation $(ry'')''-py=0$ which is a special case of (1.3) with $n=4$, $p_3=r$, and $p_i=1$ if $i\neq 3$.

In this paper we establish the existence of $\eta_k(a)$ whenever $N(k, a)$ is not empty, and we consider the properties of η_k as a function of a. We also investigate the properties of the solutions which are extremal for $\eta_k(a)$, in the following sense. A solution y of (1.3) will be called extremal if y has a zero at a, a zero at $\eta_k(a)$ and $n-1+k$ zeros in $[a, \eta_k(a)]$.

We can now state the main results.

THEOREM 1. *If y is an extremal solution for $\eta_k(a)$, then y has even order zeros at a and at $\eta_k(a)$; y has exactly $n-1+k$ zeros in $[a, \eta_k(a)]$, with exactly $k-1$ odd order zeros in $(a, \eta_k(a))$; y is never zero in $(0, a)$ or in $(\eta_k(a), \infty)$.*

THEOREM 2. *As a function of a, η_k is a strictly increasing continuous function whose domain is of the form $[0, b)$, or $[0, \infty)$.*

In order to establish Theorem 1, we will require the following results. For notational purposes, if y is a solution with $n-1+k$ zeros at r points $x_1<\cdots<x_r$ we will denote by $m(x_i)$ the multiplicity of the zero of y at x_i. Define the number

$$M(y) = \sum_{i\in I} m(x_i) + \sum_{i\in J} [m(x_i) - 1] \tag{1.5}$$

where

$$I = \{i : m(x_i) \text{ is even}\} \quad \text{and} \quad J = \{i : m(x_i) \text{ is odd}\}.$$

LEMMA 1. *If $N(m, a)$ is nonempty, then for each $k=1, \cdots, m$, there exists a kth conjugate point $\eta_k(a)$ and a nontrivial solution y_k having the following properties.*

(i) *$a<\eta_k(a)\leq\eta_{k+1}(a)$, for $k=1, \cdots, m-1$.*

(ii) *y_k has at least $n-1+k$ zeros in $[a, \eta_k(a)]$.*

(iii) *No nontrivial solution of* (1.3) *having a zero at a has more than* $n-2+k$ *zeros in* $[a, \eta_k(a))$.

For the proof, we observe that if $N(m, a)$ is nonempty then $N(k, a)$ is nonempty for $k \leqq m$, and if $x_0 \in N(m, a)$ then there exist $x_1 \leqq x_2 \leqq \cdots \leqq x_k \leqq x_0$ such that $x_i \in N(i, a)$. Thus $\eta_k(a)$ exists for each $k<m$ and $\eta_k(a) \leqq \eta_{k+1}(a)$, establishing part (i). Part (iii) is a direct consequence of the definition of $\eta_k(a)$. If $N(k, a)$ is finite, or if there exists $\varepsilon>0$ such that the intersection of $N(k, a)$ with $(\eta_k(a), \eta_k(a)+\varepsilon)$ is empty, then (ii) is immediate. Otherwise we observe that there exists a sequence $\{x_i\}$ converging monotonically to $\eta_k(a)$ and a sequence of solutions $\{y_i\}$ having zeros at a and x_i with at least $n-1+k$ zeros in $[a, x_i]$. With no loss of generality, we may assume that y_i has zeros at $a=t_{i_1}<\cdots<t_{i_r}=x_i$ and that there are integers $m_1, \cdots, m_r$ with $m_1+\cdots+m_r \geqq n-1+k$ such that $m(t_{i_j})=m_j$ for all $j=1, \cdots, r$, and all i. Normalizing each solution y_i, we may apply standard compactness arguments to obtain a nontrivial solution y of (1.3) and a subsequence $\{y_{i_j}\}$ of $\{y_i\}$ such that $L_q y_{i_j}$ converges uniformly to $L_q y$, for each $q=0, \cdots, n-1$, on $[a, x_1]$. Since limit points of the zeros $L_q y_{i_j}$ are zeros of $L_q y$ we have that y must have at least $n-1+k$ zeros in $[a, \eta_k(a)]$ with a zero at a and a zero at $\eta_k(a)$.

The following lemma, stated without proof, is due to Levin [**5**].

LEMMA 2. *There does not exist a nontrivial solution of* (1.3) *satisfying the following boundary conditions at* $x_1<\cdots<x_r$.

$$L_i y(x_j)=0, \quad i=0, \cdots, m(x_j)-1, \quad j=1, \cdots, r$$
if $m(x_i)$ *and* $m(x_r)$ *are odd and*
$m(x_i)$ *is even for* $i=2, \cdots, r-1$.

It is an immediate consequence of Lemma 2 that if y is a nontrivial solution of (1.3) with $n-1+k$ zeros at the points $x_1<\cdots<x_r$ then $M(y) \leqq n$, the number of odd order zeros of y must exceed $k-2$ and will equal $k-1$ only if $m(x_1)$ and $m(x_r)$ are even.

LEMMA 3. *If y is a nontrivial solution of* (1.3) *having* $n-1+k$ *zeros at* $x_1<\cdots<x_r$ *such that* $M(y)<n$ *then there exists a nontrivial solution of* (1.3) *with* $n-1+k$ *zeros on* $[x_1, x_r)$.

There are four cases to the proof, depending on whether the zeros at x_1 and x_r are of even or odd multiplicity. Each case is treated in a similar fashion, so we will demonstrate the case in which $m(x_1)$ is even and $m(x_r)$ is odd. Let $y_1, \cdots, y_n$ be a fundamental set of solutions of (1.3). By Lemma

2, the matrix

$$Y(\varepsilon) = \begin{bmatrix} L_0 y_1(x_1) & L_0 y_2(x_1) & \cdots & L_0 y_n(x_1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_{q_1} y_1(x_1) & L_{q_1} y_2(x_1) & \cdots & L_{q_1} y_n(x_1) \\ L_0 y_1(x_2) & L_0 y_2(x_2) & \cdots & L_0 y_n(x_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_0 y_1(x_r - \varepsilon) & L_0 y_2(x_r - \varepsilon) & \cdots & L_0 y_n(x_r - \varepsilon) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ L_{q_r} y_1(x_r - \varepsilon) & L_{q_r} y_2(x_r - \varepsilon) & \cdots & L_{q_r} y_n(x_r - \varepsilon) \end{bmatrix}$$

is nonsingular for all $\varepsilon > 0$ sufficiently small, where $q_1 = m(x_1)$; and for $i = 2, \cdots, r-1$, $q_i = m(x_i) - 1$ if $m(x_i)$ is even, $q_i = m(x_i) - 2$ if $m(x_i)$ is odd and $m(x_i) \geqq 3$, and $q_r = n - M(y) - 2$. Then for each $\varepsilon > 0$ there is a nontrivial solution vector $c(\varepsilon) = \mathrm{col}(c_1(\varepsilon), \cdots, c_n(\varepsilon))$ of $Y(\varepsilon)c(\varepsilon) = Y(0)c$ where $c = \mathrm{col}(c_1, \cdots, c_n)$ and $y = \sum_{i=1}^n c_i y_i$ is the solution with $n-1+k$ zeros in $[x_1, x_r]$. Letting $y(x, \varepsilon) = \sum_{i=1}^n c_i(\varepsilon) y_i(x)$, we have that as $\varepsilon \to 0$, $L_j y(x, \varepsilon)$ converges uniformly to $L_j y^*(x)$ for $j = 0, \cdots, n-1$ where y^* is a nontrivial solution of (1.3). Since $y - y^*$ satisfies the boundary conditions of Lemma 2, it must be the case that $y(x) = y^*(x)$ for all x. Now $y(x, \varepsilon)$ has an even order zero at each of the points $x_2, \cdots, x_{r-1}$, and hence for ε sufficiently small, $y(x, \varepsilon)$ must change sign near each odd order zero of y. A simple count establishes that $y(x, \varepsilon)$ must have $n-1+k$ zeros in $[x_1, x_r - \varepsilon]$.

COROLLARY 3.1. *If y is an extremal solution of* (1.3) *for $\eta_k(a)$, then the zeros at a and $\eta_k(a)$ are of even multiplicity.*

If either a or $\eta_k(a)$ is of odd multiplicity, then $M(y) < n$, which contradicts the fact that y is extremal.

COROLLARY 3.2. *If y is an extremal solution for $\eta_k(a)$, then y has exactly $n-1+k$ zeros in $[a, \eta_k(a)]$.*

If y has $n-1+r$ zeros in $[a, \eta_k(a)]$, and if $r > k$, then $m(\eta_k(a)) > r-k$. Using the techniques of Lemma 3 with the multiplicity $m(\eta_k(a)) - 1$ at $\eta_k(a)$ yields a solution with $n-2+r$ zeros in $[a, \eta_k(a) - \varepsilon]$.

If an extremal solution for $\eta_k(a)$ has m odd order zeros, then clearly $M(y)+m=n-1+k$, so that $m=k-1$. If y has a zero in either $(0, a)$ or $(\eta_k(a), \infty)$, then y satisfies the boundary conditions of Lemma 2, yielding a contradiction. This completes the proof of Theorem 1.

LEMMA 4. *If $\eta_k(b)<\infty$, then there exists $\delta>0$ such that for a in $(b-\delta, b+\delta)$, $\eta_k(a)<\infty$.*

Let y be an extremal solution for $\eta_k(b)$ with zeros at $b=x_1<\cdots<x_r=\eta_k(b)$. Then, for each ε, sufficiently small, we define $y(x, \varepsilon)$ to be a solution of (1.3) having zeros of multiplicity $m(x_i)-1$ at x_i if $i=1$, r or if $m(x_i)$ is odd, a zero of multiplicity $m(x_i)$ at x_i if $m(x_i)$ is even, $1<i<r$, and zero at $b+\varepsilon$. Then we may write

$$y(x, \varepsilon) = \sum_{i=1}^{n} c_i(\varepsilon) y_i(x)$$

where $\{y_1, \cdots, y_n\}$ is a fundamental set of solutions of (1.3), and with no loss of generality,

$$\sum_{i=1}^{n} c_i(\varepsilon)^2 = 1.$$

Then as $\varepsilon\to 0$, $L_j y(x, \varepsilon)$ converges uniformly to $L_j y^*(x)$, where y^* is a nontrivial solution of (1.3) satisfying

$$y^*(x) = \sum_{i=1}^{n} c_i y_i(x), \qquad \sum_{i=1}^{n} c_i^2 = 1.$$

If $y^*(x)\neq ky(x)$ for all x, then there is a nontrivial linear combination of y^* and y satisfying the boundary conditions in Lemma 2, which is a contradiction. Then there is a $\delta>0$ such that if $|\varepsilon|<\delta$, $y(x, \varepsilon)$ changes sign near each odd order zero of y^* and near $\eta_k(b)$ since $y(x, \varepsilon)$ has an odd order zero at $\eta_k(b)$ and y^* has an even order zero there. A simple count yields that $y(x, \varepsilon)$ has $n-1+k$ zeros in $[b+\varepsilon, \eta_k(b)+\delta]$, and this completes the proof, since a similar argument holds for $b-\varepsilon$.

If we define $y(x, \varepsilon)$ as in Lemma 4 at the points $x_1, \cdots, x_r$ and require that $y(\eta_k(b)-\varepsilon, \varepsilon)=0$ rather than $y(b-\varepsilon)=0$, for $\varepsilon>0$ then we obtain the following.

LEMMA 5. *If $\eta_k(b)$ exists, then there is a sequence $\{a_i\}$ converging to b in $(b-\delta, b)$ such that $\eta_k(a_i)<\eta_k(b)$ for each i.*

From the preceding discussion, we have for each $\varepsilon>0$, sufficiently small, that there exists a nontrivial solution $y(x, \varepsilon)$ of (1.3) with $n-1+k$ zeros in $[b-\varepsilon, \eta_k(b)]$. From Theorem 1, $y(x, \varepsilon)$ is not an extremal solution for $\eta_k(b-\varepsilon)$, hence $\eta_k(b-\varepsilon)<\eta_k(b)$.

COROLLARY 5.1. *If* $\eta_k(x) < \infty$ *for all* $x \in [a, b]$ *then* $\eta_k(a) < \eta_k(b)$.

Suppose to the contrary that $\eta_k(a) \geqq \eta_k(b)$. For some $\varepsilon > 0$, $\eta_k(b-\varepsilon) < \eta_k(b)$, $b-\varepsilon > a$ and $S = \{x > a : \eta_k(x) < \eta_k(b-\varepsilon)\}$ is nonempty, and hence if $d = \inf S$, then $\eta_k(d) = \eta_k(b-\varepsilon)$. If $a < d$, then $\eta_k(x) \geqq \eta_k(b-\varepsilon) \geqq \eta_k(d)$ for all x in (a, d), contradicting Lemma 6. Thus $a = d$. Let x_i be a sequence in S converging monotonically to a. Then arguments of Lemma 1 yield a sequence of extremal solutions y_i converging to a solution y of (1.3) having $n-1+k$ zeros in $[a, \eta_k(b-\varepsilon)]$. But this contradicts the definition of $\eta_k(a)$ since $\eta_k(a) \geqq \eta_k(b) > \eta_k(b-\varepsilon)$.

COROLLARY 5.2. *If* $\eta_k(b) < \infty$, *then* $\eta_k(x) < \infty$ *for all* $x \leqq b$.

If, to the contrary, there exists an $x < b$ for which $N(x, k)$ is empty, let $a = \sup\{x < b : N(x, k)$ is empty$\}$. Then $a < b$, and $\eta_k(x) < \infty$ for all x in (a, b). From Corollary 6.1, there is a sequence $\{a_i\}$ in (a, b) such that $a_{i+1} < a_i$, $\eta_k(a_{i+1}) < \eta_k(a_i)$ and a_i converges to a as i tends to ∞. Then the techniques of Lemma 1 yield a solution y of (1.3) such that $y(a) = 0$ and y has $n-1+k$ zeros on $[a, \eta_k(b)]$, contradicting Lemma 4.

COROLLARY 5.3. η_k *is continuous*.

If as $x \to a-0$, $\eta_k(x) \to L < \eta_k(a)$, then the arguments of Lemma 1 yield a nontrivial solution of (1.3) with $n-1+k$ zeros in $[a, \eta_k(a)]$ since η_k is increasing. If as $x \to a+0$, $\eta_k(x) \to L > \eta_k(a)$, let $\delta = \frac{1}{2}(L - \eta_k(a))$. By Lemma 4, there exists $\varepsilon > 0$ and a solution of $y(x, \varepsilon)$ having $n-1+k$ zeros in $[a+\varepsilon, \eta_k(a)+\delta]$. This contradicts the fact that η_k is increasing.

Corollaries 5.1, 5.2, and 5.3 complete the proof of Theorem 2.

REFERENCES

1. J. H. Barrett, *Oscillation theory of ordinary linear differential equations*, Advances in Math. **3** (1969), 415–509. MR **41** #2113.

2. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955. MR **16**, 1022.

3. G. W. Johnson, *The zeros of solutions of an even order quasi differential equation*, J. Analyse Math. (to appear).

4. W. Leighton and Z. Nehari, *On the oscillation of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. **89** (1958), 325–377. MR **21** #1429.

5. A. Ju. Levin, *Some problems bearing on the oscillation of solutions of linear differential equations*, Dokl. Akad. Nauk SSSR **148** (1963), 512–515 = Soviet Math. Dokl. **4** (1963), 121–124. MR **26** #3972.

6. Z. Nehari, *Disconjugate linear differential operators*, Trans. Amer. Math. Soc. **129** (1967), 500–516. MR **36** #2860.

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