THE kTH CONJUGATE POINT_FUNCTION FOR AN EVEN ORDER LINEAR DIFFERENTIAL EQUATION

GEORGE W. JOHNSON

Abstract. For an even order, two term equation \( L_n y = p(x) y \), \( p(x) > 0 \), \( x \in [0, \infty) \), the \( k \)th conjugate point function \( \eta_k(a) \) is defined and is shown to be a strictly increasing continuous function with domain \([0, b)\) or \([0, \infty)\). Extremal solutions are defined as nontrivial solutions with \( n-1+k \) zeros on \([a, \eta_k(a)]\), and are shown to have exactly \( n-1+k \) zeros, with even order zeros at \( a \) and \( \eta_k(a) \) and exactly \( k-1 \) odd order zeros in \((a, \eta_k(a))\), thus establishing that \( \eta_k(a) < \eta_{k+1}(a) \).

The differential equation considered in this paper is defined as follows: Let \( p_1, \ldots, p_{n+1} \) be positive continuous functions defined on \([0, \infty)\), and let \( A_0 \) denote the set of all continuous functions defined on \([0, \infty)\). For \( y \) in \( A_0 \), define

\[ L_0 y = p_1 y. \]

Assume that \( A_i \) and \( L_i y \) have been defined for \( i \leq k-1 \), and let \( A_k \) denote the set of all functions \( y \) for which \( L_{k-1} y \) has a continuous derivative on \([0, \infty)\). For \( y \) in \( A_k \), define

\[ L_k y = p_{k+1}(L_{k-1} y)'. \]

The differential equation with which we are concerned is

\[ L_n y = p y, \]

where \( p \) is a positive continuous function defined on \([0, \infty)\), and \( n \geq 4 \) is an even integer (cf. [3], [5], [6]).

From (1.2) and (1.3) it follows that \( L_n \) has the factored form

\[ L_n y = p_{n+1}(p_n \cdots p_2(p_1 y)' \cdots ')'. \]

For \( y \) in \( A_n \), the function \( L_i y \) is said to have a zero of multiplicity \( k \) at \( x=a \) if

\[ L_i y(a) = \cdots = L_{k-1+i} y(a) = 0 \quad \text{and} \quad L_{k+i} y(a) \neq 0. \]
Since the functions $p_1, \cdots, p_{n+1}$ are positive, it follows that if $L_{k-1}y(a) = L_{k-1}y(b) = 0$ for some $a < b$, then there is a number $c$ in $(a, b)$ for which $L_ky(c) = 0$. Moreover, if $L_iy$ has a zero of multiplicity $k$ at $x = a$, then $L_iy$ changes sign at $x = a$ if and only if $k$ is odd.

Denote by $N(k, a)$ the set of all points $x > a$ for which there is a nontrivial solution of (1.3), with zeros at $a$ and $x$, having at least $n-1+k$ zeros in $[a, x]$. The $k$th conjugate point of $a$, denoted by $\eta_k(a)$, is defined to be the infimum of the set $N(k, a)$; if $N(k, a)$ is empty, then $\eta_k(a) = +\infty$.

Leighton and Nehari [4] studied the functions $\eta_k$ extensively for the equation $(ry')' - py = 0$ which is a special case of (1.3) with $n = 4$, $p_3 = r$, and $p_1 = 1$ if $i \neq 3$.

In this paper we establish the existence of $\eta_k(a)$ whenever $N(k, a)$ is not empty, and we consider the properties of $\eta_k$ as a function of $a$. We also investigate the properties of the solutions which are extremal for $\eta_k(a)$, in the following sense. A solution $y$ of (1.3) will be called extremal if $y$ has a zero at $a$, a zero at $\eta_k(a)$ and $n-1+k$ zeros in $[a, \eta_k(a)]$.

We can now state the main results.

**Theorem 1.** If $y$ is an extremal solution for $\eta_k(a)$, then $y$ has even order zeros at $a$ and at $\eta_k(a)$; $y$ has exactly $n-1+k$ zeros in $[a, \eta_k(a)]$, with exactly $k-1$ odd order zeros in $(a, \eta_k(a))$; $y$ is never zero in $(0, a)$ or in $(\eta_k(a), \infty)$.

**Theorem 2.** As a function of $a$, $\eta_k$ is a strictly increasing continuous function whose domain is of the form $[0, b)$ or $[0, \infty)$.

In order to establish Theorem 1, we will require the following results. For notational purposes, if $y$ is a solution with $n-1+k$ zeros at $r$ points $x_1 < \cdots < x_r$ we will denote by $m(x_i)$ the multiplicity of the zero of $y$ at $x_i$. Define the number

\[
M(y) = \sum_{i \in I} m(x_i) + \sum_{i \in J} [m(x_i) - 1]
\]

where

$I = \{i : m(x_i) \text{ is even}\} \quad \text{and} \quad J = \{i : m(x_i) \text{ is odd}\}$

**Lemma 1.** If $N(m, a)$ is nonempty, then for each $k = 1, \cdots, m$, there exists a $k$th conjugate point $\eta_k(a)$ and a nontrivial solution $y_k$ having the following properties.

(i) $a < \eta_k(a) \leq \eta_{k+1}(a)$, for $k = 1, \cdots, m-1$.
(ii) $y_k$ has at least $n-1+k$ zeros in $[a, \eta_k(a)]$.
(iii) No nontrivial solution of (1.3) having a zero at a has more than \( n-2+k \) zeros in \([a, \eta_k(a)]\).

For the proof, we observe that if \( N(m, a) \) is nonempty then \( N(k, a) \) is nonempty for \( k \leq m \), and if \( x_0 \in N(m, a) \) then there exist \( x_1 \leq x_2 \leq \cdots \leq x_k \leq x_0 \) such that \( x_t \in N(i, a) \). Thus \( \eta_k(a) \) exists for each \( k < m \) and \( \eta_k(a) \leq \eta_{k+1}(a) \), establishing part (i). Part (iii) is a direct consequence of the definition of \( \eta_k(a) \). If \( N(k, a) \) is finite, or if there exists \( \varepsilon > 0 \) such that the intersection of \( N(k, a) \) with \( (\eta_k(a), \eta_k(a) + \varepsilon) \) is empty, then (ii) is immediate. Otherwise we observe that there exists a sequence \( \{x_t\} \) converging monotonically to \( \eta_k(a) \) and a sequence of solutions \( \{y_t\} \) having zeros at \( a \) and \( x_t \) with at least \( n-1+k \) zeros in \([a, x_t]\). With no loss of generality, we may assume that \( y_t \) has zeros at \( a=t_{i_1} < \cdots < t_{i_r} = x_t \) and that there are integers \( m_{i_1}, \cdots, m_r \) with \( m_1 + \cdots + m_r = n-1+k \) such that \( m(t_{i_j}) = m_j \) for all \( j=1, \cdots, r \), and all \( i \). Normalizing each solution \( y_t \), we may apply standard compactness arguments to obtain a nontrivial solution \( y \) of (1.3) and a sequence of solutions \( \{y_t\} \) such that \( L_q y_t \) converges uniformly to \( L_q y \), for each \( q=0, \cdots, n-1 \), on \([a, x_t]\). Since limit points of the zeros \( L_q y_t \) are zeros of \( L_q y \) we have that \( y \) must have at least \( n-1+k \) zeros in \([a, \eta_k(a)]\) with a zero at \( a \) and a zero at \( \eta_k(a) \).

The following lemma, stated without proof, is due to Levin [5].

**Lemma 2.** There does not exist a nontrivial solution of (1.3) satisfying the following boundary conditions at \( x_1 < \cdots < x_r \).

\[
L_i y(x_j) = 0, \quad i = 0, \cdots, m(x_i) - 1, \quad j = 1, \cdots, r
\]

if \( m(x_i) \) and \( m(x_r) \) are odd and

\[
m(x_i) \text{ is even for } i = 2, \cdots, r-1.
\]

It is an immediate consequence of Lemma 2 that if \( y \) is a nontrivial solution of (1.3) with \( n-1+k \) zeros at the points \( x_1 < \cdots < x_r \), then \( M(y) = n \), the number of odd order zeros of \( y \) must exceed \( k-2 \) and will equal \( k-1 \) only if \( m(x_1) \) and \( m(x_r) \) are even.

**Lemma 3.** If \( y \) is a nontrivial solution of (1.3) having \( n-1+k \) zeros at \( x_1 < \cdots < x_r \) such that \( M(y) < n \) then there exists a nontrivial solution of (1.3) with \( n-1+k \) zeros on \([x_1, x_r]\).

There are four cases to the proof, depending on whether the zeros at \( x_1 \) and \( x_r \) are of even or odd multiplicity. Each case is treated in a similar fashion, so we will demonstrate the case in which \( m(x_1) \) is even and \( m(x_r) \) is odd. Let \( y_1, \cdots, y_n \) be a fundamental set of solutions of (1.3). By Lemma
2, the matrix

\[
Y(e) = \begin{bmatrix}
L_0y_1(x_1) & L_0y_2(x_1) & \cdots & L_0y_n(x_1) \\
\vdots & \vdots & & \vdots \\
L_0y_1(x_2) & L_0y_2(x_2) & \cdots & L_0y_n(x_2) \\
L_0y_1(x_{r-\varepsilon}) & L_0y_2(x_{r-\varepsilon}) & \cdots & L_0y_n(x_{r-\varepsilon}) \\
\vdots & \vdots & & \vdots \\
L_0y_1(x_1) & L_0y_2(x_1) & \cdots & L_0y_n(x_1)
\end{bmatrix}
\]

is nonsingular for all \(\varepsilon > 0\) sufficiently small, where \(q_i = m(x_i)\); and for \(i = 2, \ldots, r-1\), \(q_i = m(x_i) - 1\) if \(m(x_i)\) is even, \(q_i = m(x_i) - 2\) if \(m(x_i)\) is odd and \(m(x_i) \geq 3\), and \(q_r = n - M(y) - 2\). Then for each \(\varepsilon > 0\) there is a nontrivial solution vector \(c(e) = (c_1(e), \ldots, c_n(e))\) of \(Y(e)c(e) = Y(0)c\) where \(c = (c_1, \ldots, c_n)\) and \(y = \sum_{i=1}^{n} c_i y_i\) is the solution with \(n-1+k\) zeros in \([x_1, x_r]\). Letting \(y(x, e) = \sum_{i=1}^{n} c_i e(y_i(x))\), we have that as \(\varepsilon \to 0\), \(L_j y(x, e)\) converges uniformly to \(L_j y^*(x)\) for \(j = 0, \ldots, n-1\) where \(y^*\) is a nontrivial solution of (1.3). Since \(y - y^*\) satisfies the boundary conditions of Lemma 2, it must be the case that \(y(x) = y^*(x)\) for all \(x\). Now \(y(x, e)\) has an even order zero at each of the points \(x_2, \ldots, x_{r-1}\), and hence for \(\varepsilon\) sufficiently small, \(y(x, e)\) must change sign near each odd order zero of \(y\). A simple count establishes that \(y(x, e)\) must have \(n-1+k\) zeros in \([x_1, x_r-e]\).

**Corollary 3.1.** If \(y\) is an extremal solution of (1.3) for \(\eta_k(a)\), then the zeros at \(a\) and \(\eta_k(a)\) are of even multiplicity.

If either \(a\) or \(\eta_k(a)\) is of odd multiplicity, then \(M(y) < n\), which contradicts the fact that \(y\) is extremal.

**Corollary 3.2.** If \(y\) is an extremal solution for \(\eta_k(a)\), then \(y\) has exactly \(n-1+k\) zeros in \([a, \eta_k(a)]\).

If \(y\) has \(n-1+r\) zeros in \([a, \eta_k(a)]\), and if \(r > k\), then \(m(\eta_k(a)) > r-k\). Using the techniques of Lemma 3 with the multiplicity \(m(\eta_k(a)) - 1\) at \(\eta_k(a)\) yields a solution with \(n-2+r\) zeros in \([a, \eta_k(a)-\varepsilon]\).
If an extremal solution for \( \eta_k(a) \) has \( m \) odd order zeros, then clearly 
\[
M(y) + m = n - 1 + k,
\]
so that \( m = k - 1 \). If \( y \) has a zero in either \((0, a)\) or 
\((\eta_k(a), \infty)\), then \( y \) satisfies the boundary conditions of Lemma 2, yielding 
a contradiction. This completes the proof of Theorem 1.

**Lemma 4.** If \( \eta_k(b) < \infty \), then there exists \( \delta > 0 \) such that for \( a \) in 
\((b - \delta, b + \delta)\), \( \eta_k(a) < \infty \).

Let \( y \) be an extremal solution for \( \eta_k(b) \) with zeros at 
\( b = x_1 < \cdots < x_r = \eta_k(b) \). Then, for each \( \epsilon \), sufficiently small, we define \( y(x, \epsilon) \) to be 
a solution of (1.3) having zeros of multiplicity \( m(x_i) = 1 \) at \( x_i \) if \( i = 1, r \) or if \( m(x_i) \) is odd, a zero of multiplicity \( m(x_i) \) at \( x_i \) if \( m(x_i) \) is even, 
\( 1 < i < r \), and zero at \( b + \epsilon \). Then we may write

\[
y(x, \epsilon) = \sum_{i=1}^{n} c_i(\epsilon)y_i(x)
\]

where \( \{y_1, \ldots, y_n\} \) is a fundamental set of solutions of (1.3), and with no loss of generality,

\[
\sum_{i=1}^{n} c_i(\epsilon)^2 = 1.
\]

Then as \( \epsilon \to 0 \), \( L_jy(x, \epsilon) \) converges uniformly to \( L_jy^*(x) \), where \( y^* \) is 
a nontrivial solution of (1.3) satisfying

\[
y^*(x) = \sum_{i=1}^{n} c_iy_i(x), \quad \sum_{i=1}^{n} c_i^2 = 1.
\]

If \( y^*(x) \neq ky(x) \) for all \( x \), then there is a nontrivial linear combination 
of \( y^* \) and \( y \) satisfying the boundary conditions in Lemma 2, which is a 
contradiction. Then there is a \( \delta > 0 \) such that if \( |\epsilon| < \delta \), \( y(x, \epsilon) \) changes 
sign near each odd order zero of \( y^* \) and near \( \eta_k(b) \) since \( y(x, \epsilon) \) has an 
odd order zero at \( \eta_k(b) \) and \( y^* \) has an even order zero there. A simple 
count yields that \( y(x, \epsilon) \) has \( n - 1 + k \) zeros in \([b + \epsilon, \eta_k(b) + \delta]\), and this 
completes the proof, since a similar argument holds for \( b - \epsilon \).

If we define \( y(x, \epsilon) \) as in Lemma 4 at the points \( x_1, \ldots, x_r \) and require 
that \( y(\eta_k(b) - \epsilon, \epsilon) = 0 \) rather than \( y(b - \epsilon) = 0 \), for \( \epsilon > 0 \) then we obtain the 
following.

**Lemma 5.** If \( \eta_k(b) \) exists, then there is a sequence \( \{a_i\} \) converging to \( b \) 
in \((b - \delta, b)\) such that \( \eta_k(a_i) < \eta_k(b) \) for each \( i \).

From the preceding discussion, we have for each \( \epsilon > 0 \), sufficiently 
small, that there exists a nontrivial solution \( y(x, \epsilon) \) of (1.3) with \( n - 1 + k \) 
zeros in \([b - \epsilon, \eta_k(b)]\). From Theorem 1, \( y(x, \epsilon) \) is not an extremal solution 
for \( \eta_k(b - \epsilon) \), hence \( \eta_k(b - \epsilon) < \eta_k(b) \).
Corollary 5.1. If $\eta_k(x) < \infty$ for all $x \in [a, b]$ then $\eta_k(a) < \eta_k(b)$.

Suppose to the contrary that $\eta_k(a) \geq \eta_k(b)$. For some $\varepsilon > 0$, $\eta_k(b-\varepsilon) < \eta_k(b)$, $b-\varepsilon > a$ and $S = \{x > a : \eta_k(x) < \eta_k(b-\varepsilon)\}$ is nonempty, and hence if $d = \inf S$, then $\eta_k(d) = \eta_k(b-\varepsilon)$. If $a < d$, then $\eta_k(x) \geq \eta_k(b-\varepsilon) \geq \eta_k(d)$ for all $x$ in $(a, d)$, contradicting Lemma 6. Thus $a = d$. Let $x_i$ be a sequence in $S$ converging monotonically to $a$. Then arguments of Lemma 1 yield a sequence of extremal solutions $y_i$ converging to a solution $y$ of (1.3) having $n-1+k$ zeros in $[a, \eta_k(b-\varepsilon)]$. But this contradicts the definition of $\eta_k(a)$ since $\eta_k(a) > \eta_k(b) > \eta_k(b-\varepsilon)$.

Corollary 5.2. If $\eta_k(b) < \infty$, then $\eta_k(x) < \infty$ for all $x \leq b$.

If, to the contrary, there exists an $x < b$ for which $N(x, k)$ is empty, let $a = \sup\{x < b : N(x, k) \text{ is empty}\}$. Then $a < b$, and $\eta_k(x) < \infty$ for all $x$ in $(a, b)$. From Corollary 6.1, there is a sequence $\{a_i\}$ in $(a, b)$ such that $a_{i+1} < a_i$, $\eta_k(a_{i+1}) < \eta_k(a_i)$, and $a_i$ converges to $a$ as $i$ tends to $\infty$. Then the techniques of Lemma 1 yield a solution $y$ of (1.3) such that $y(a) = 0$ and $y$ has $n-1+k$ zeros on $[a, \eta_k(b)]$, contradicting Lemma 4.

Corollary 5.3. $\eta_k$ is continuous.

If as $x \to a^-$, $\eta_k(x) \to L < \eta_k(a)$, then the arguments of Lemma 1 yield a nontrivial solution of (1.3) with $n-1+k$ zeros in $[a, \eta_k(a)]$ since $\eta_k$ is increasing. If as $x \to a^+$, $\eta_k(x) \to L > \eta_k(a)$, let $\delta = \frac{1}{2}(L - \eta_k(a))$. By Lemma 4, there exists $\varepsilon > 0$ and a solution of $y(x, \varepsilon)$ having $n-1+k$ zeros in $[a+\varepsilon, \eta_k(a)+\delta]$. This contradicts the fact that $\eta_k$ is increasing.

Corollaries 5.1, 5.2, and 5.3 complete the proof of Theorem 2.

References


Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208