Λ(p) SETS AND THE EXACT MAJORANT PROPERTY

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ABSTRACT. Let Γ be a discrete abelian group. We prove that if 2 < p < ∞ and E ⊂ Γ, then (E, p) has the exact majorant property if and only if E is a Λ(p) set.

Let G be a compact abelian group with dual group Γ. If 1 ≤ r < p, let $L_r^p(G) = \{ f : f \in L^p(G), f(y) = 0, \gamma \not\in E \}$. If $L_r^p(G) = L_r^p(G)$ for a set E, we say E is a Λ(p) set. It follows that if $L_r^p(G) = L_r^p(G)$ for some $r$, 1 ≤ r < p, then $L_r^p(G) = L_r^p(G) = L_r^p(G)$ (cf. [6, 37.7]). If $f, g \in L^p(G)$ and $g = \hat{f}$ we say as in [5] that g is the exact majorant of f. We say (E, p) has the exact majorant property if whenever $f \in L_r^p(G)$ then $\frac{1}{|f|} \in (L_r^p(G))^\wedge$, where $(L_r^p(G))^\wedge = \{ f : f \in L_r^p(G) \}$.

If $p = 2$ and $E \subset \Gamma$, then $(E, 2)$ always has the exact majorant property since $L^2(G)$ does. The following theorem and remark show that if $2 < p \leq \infty$ then $(E, p)$ has the exact majorant property only for special sets E.

Theorem. Suppose $2 < p < \infty$; then $(E, p)$ has the exact majorant property if and only if E is a Λ(p) set.

Proof. Suppose E is a Λ(p) set. If $f \in L_r^p(G)$ then $f \in L_r^p(G)$, since $L_r^p(G) \subset L_r^p(G)$ for $p > 2$. So $\hat{f} \in (L_r^p(G))^\wedge$ and $\frac{1}{|f|} \in (L_r^p(G))^\wedge$. But E is a Λ(p) set so $L_r^p(G) = L_r^p(G)$ and $\frac{1}{|f|} \in (L_r^p(G))^\wedge = (L_r^p(G))^\wedge$.

Conversely suppose $(E, p)$ has the exact majorant property. We wish to show $L_r^p(G) = L_r^p(G)$. But $L_r^p(G) \subset L_r^p(G)$, so it is sufficient to show $L_r^p(G) \subset L_r^p(G)$.

Suppose $f \in L_r^p(G)$. Define $f_1$ and $f_2$ in $L_r^p(G)$ as follows:

$$f_1(\gamma) = \frac{1}{2}(\text{Re} \hat{f}(\gamma)) + \text{Re} \hat{f}(\gamma), \quad f_2(\gamma) = \frac{1}{2}(\text{Re} \hat{f}(\gamma)) - \text{Re} \hat{f}(\gamma).$$

Define $f_3$ and $f_4$ similarly with Re $\hat{f}$ replaced by Im $\hat{f}$. Then clearly $f_i \in L_r^p(G)$ for $1 \leq i \leq 4$ and

$$f = f_1 - f_2 + i(f_3 - f_4). \quad (1)$$

Consider $f_i$, by [3, 14.3.2] or [6, 36.5], there exists a choice of numbers $c_i$ of absolute value 1 such that if

$$h_i(x) \sim \sum c_i f_i(\gamma)(x, \gamma),$$

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then $h_1 \in L_E^p(G)$. But $(E, p)$ has the exact majorant property, so $|h_1| \in L_E^p(G)$.

Similarly $f_j \in L_E^p(G)$ for $2 \leq j \leq 4$. By equation (1), $f \in L_E^p(G)$.

**Remarks.** This theorem was proved for the special case when $p$ is an even integer $>2$ by Bachelis [1]. The proof actually gives the following: If $p$ is an even integer $>2$, then $E$ is a $\Lambda(p)$ set if and only if given $f \in L_E^p(G)$ there exists $g \in L_E^p(G)$ with $|f| \leq g$. (See [1, Theorem 3 and Lemma 1], or [2, Theorem 3].) The question of whether or not this characterization is also valid for $p$ not an even integer is open.

It is easy to show that $(E, \infty)$ has the exact majorant property if and only if $E$ is a Sidon set. It is known that a Sidon set is a $\Lambda(p)$ set for all $p < \infty$ [6, 37.10]. A natural question to ask is, if $(E, p)$ has the exact majorant property for all $p < \infty$, does $(E, \infty)$ have the exact majorant property? This question is answered in the negative since every infinite discrete abelian group contains a set $E$ which is $\Lambda(p)$ for all $p < \infty$ but not Sidon [4].

**Bibliography**


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