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$\Lambda(p)$ SETS AND THE EXACT MAJORANT PROPERTY

SAMUEL E. EBENSTEIN

ABSTRACT. Let Γ be a discrete abelian group. We prove that if $2<p<\infty$ and $E\subset\Gamma$, then (E,p) has the exact majorant property if and only if E is a $\Lambda(p)$ set.

Let G be a compact abelian group with dual group Γ. If $1\leqq p\leqq\infty$ and $E\subset\Gamma$, let $L^p_E(G)=\{f:f\in L^p(G),\ \hat{f}(\gamma)=0,\ \gamma\notin E\}$. If $L^1_E(G)=L^p_E(G)$ for a set E, we say E is a $\Lambda(p)$ set. It follows that if $L^r_E(G)=L^p_E(G)$ for some r, $1\leqq r<p$, then $L^1_E(G)=L^r_E(G)=L^p_E(G)$ (cf. [6, 37.7]). If $f,g\in L^p(G)$ and $\hat{g}=|\hat{f}|$ we say as in [5] that g is the exact majorant of f. We say (E,p) has the exact majorant property if whenever $f\in L^p_E(G)$ then $|\hat{f}|\in(L^p_E(G))^\wedge$, where $(L^p_E(G))^\wedge=\{\hat{f}:f\in L^p_E(G)\}$.

If $p=2$ and $E\subset\Gamma$, then $(E,2)$ always has the exact majorant property since $L^2(G)$ does. The following theorem and remark show that if $2<p\leqq\infty$ then (E,p) has the exact majorant property only for special sets E.

THEOREM. *Suppose $2<p<\infty$; then (E,p) has the exact majorant property if and only if E is a $\Lambda(p)$ set.*

PROOF. Suppose E is a $\Lambda(p)$ set. If $f\in L^p_E(G)$ then $f\in L^2_E(G)$, since $L^p_E(G)\subset L^2_E(G)$ for $p>2$. So $\hat{f}\in(L^2_E(G))^\wedge$ and $|\hat{f}|\in(L^2_E(G))^\wedge$. But E is a $\Lambda(p)$ set so $L^2_E(G)=L^p_E(G)$ and $|\hat{f}|\in(L^2_E(G))^\wedge=(L^p_E(G))^\wedge$.

Conversely suppose (E,p) has the exact majorant property. We wish to show $L^2_E(G)=L^p_E(G)$. But $L^p_E(G)\subset L^2_E(G)$, so it is sufficient to show $L^2_E(G)\subset L^p_E(G)$.

Suppose $f\in L^2_E(G)$. Define f_1 and f_2 in $L^2_E(G)$ as follows:

$$\hat{f}_1(\gamma)=\tfrac{1}{2}(|\mathrm{Re}\hat{f}(\gamma)|+\mathrm{Re}\hat{f}(\gamma)),\qquad \hat{f}_2(\gamma)=\tfrac{1}{2}(|\mathrm{Re}\hat{f}(\gamma)|-\mathrm{Re}\hat{f}(\gamma)).$$

Define f_3 and f_4 similarly with $\mathrm{Re}\hat{f}$ replaced by $\mathrm{Im}\hat{f}$. Then clearly $f_j\in L^2_E(G)$ for $1\leqq j\leqq 4$ and

$$f=f_1-f_2+i(f_3-f_4). \tag{1}$$

Consider f_1, by [3, 14.3.2] or [6, 36.5], there exists a choice of numbers c_γ of absolute value 1 such that if

$$h_1(x)\sim\sum c_\gamma\hat{f}_1(\gamma)(x,\gamma),$$

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then $h_1 \in L_E^p(G)$. But (E, p) has the exact majorant property, so $|\hat{h}_1| \in L_E^p(G)$.

Similarly $f_j \in L_E^p(G)$ for $2 \leqq j \leqq 4$. By equation (1), $f \in L_E^p(G)$.

REMARKS. This theorem was proved for the special case when p is an even integer >2 by Bachelis [**1**]. The proof actually gives the following: If p is an even integer >2, then E is a $\Lambda(p)$ set if and only if given $f \in L_E^p(G)$ there exists $g \in L_E^p(G)$ with $|\hat{f}| \leqq \hat{g}$. (See [**1**, Theorem 3 and Lemma 1], or [**2**, Theorem 3].) The question of whether or not this characterization is also valid for p not an even integer is open.

It is easy to show that (E, ∞) has the exact majorant property if and only if E is a Sidon set. It is known that a Sidon set is a $\Lambda(p)$ set for all $p < \infty$ [**6**, 37.10]. A natural question to ask is, if (E, p) has the exact majorant property for all $p < \infty$, does (E, ∞) have the exact majorant property? This question is answered in the negative since every infinite discrete abelian group contains a set E which is $\Lambda(p)$ for all $p < \infty$ but not Sidon [**4**].

BIBLIOGRAPHY

1. G. F. Bachelis, *On the ideal of unconditionally convergent Fourier series in* $L^p(G)$, Proc. Amer. Math. Soc. **27** (1971), 309–312. MR **42** #6523.

2. ———, *On the upper and lower majorant properties for* $L^p(G)$, Quart. J. Math. **24** (1973), 119–128.

3. R. E. Edwards, *Fourier series: A modern introduction*. Vol. II, Holt, Rinehart and Winston, New York, 1967. MR **36** #5588.

4. R. E. Edwards, E. Hewitt and K. A. Ross, *Lacunarity for compact groups*. I, Indiana Univ. Math. J. **21** (1972), 787–806.

5. G. H. Hardy and J. E. Littlewood, *A problem concerning majorants of Fourier series*, Quart. J. Math. **6** (1935), 304–315.

6. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. II: *Structure and analysis for compact groups analysis on locally compact Abelian groups*, Die Grundlehren der math. Wissenschaften, Band 152, Springer-Verlag, New York and Berlin, 1970. MR **41** #7378.

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202