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SHORTER NOTES

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NECESSARY AND SUFFICIENT CONDITIONS FOR CARLSON'S THEOREM FOR ENTIRE FUNCTIONS[1]

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In his thesis, F. Carlson proved the following important result (see [5]): Let f be an entire function of exponential type whose type on the imaginary axis is $C<\pi$, and suppose that $f(n)=0$ for all $n\in I=\{1,2,3,\cdots\}$; then f is identically zero. A set A of integers is said to be a Carlson set if this statement holds with I replaced by A. In the search for necessary and sufficient conditions for A to be a Carlson set, it was discovered that $D(A)=1$ is sufficient, and $UD(A)=1$ is necessary. (See [2].) Here, $D(A)$ is the (asymptotic) density and $UD(A)$ the upper density of A, defined respectively by $\lim A(n)/n$ and $\limsup A(n)/n$, where $A(n)$ is the number of integers in $A\cap[1,n]$.

In 1954, while L. A. Rubel was completing his doctoral work with me, we explored another class of Carlson sets, those having very long blocks. (See [4] and [7].) Several years later, Rubel combined a detailed estimate derived from Carleman's theorem with a tauberian theorem for logarithmic density to show that $UD(A)=1$ is also sufficient for A to be a Carlson set, thus attaining the long desired result [8].

The purpose of the present short note is to show that an elementary theorem about sets of zero density immediately reduces this Carlson problem to the study of sets with very long blocks, from which Rubel's result follows at once by a classical gap theorem for power series.

A set A is said to have very long blocks if A contains an infinite number of intervals of the form $I_j=\{\text{all } k \text{ with } n_j\leq k\leq n_j'\}$ where $n_j'/n_j\to\infty$. The crucial observation is the following.

THEOREM 1. *A set A obeys $UD(A)=1$ if and only if there is a set S of zero density such that $A\cup S$ has very long blocks.*

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PROOF. If $UD(A)=1$, so that $\limsup A(n)/n=1$, then one may choose an increasing sequence n_k so that $A(n_k)/n_k>1-(1/k)^3$ while $n_{k+1}/n_k>1+k^2$. Let $J_k=\{\text{all } m \text{ with } (1/k)n_k\leqq m\leqq n_k\}$. These blocks are all disjoint. Put $S=\bigcup\{J_k\cap A'\}$ where A' is the complement of A in I; the set $A\cup S$ has very long blocks since it contains the union of all the J_k. To see that $D(S)=0$, suppose that n belongs to some J_k. Then one sees that $S(n)/n<k^{-2}$. If instead, n falls between blocks J_k and J_{k+1}, then $S(n)/n<k^{-1}$. Hence $\lim S(n)/n=0$. (The other direction of the implication is immediate.)

We next state a gap theorem obtained in 1926 by Ostrowski. (See [**6**], [**7**], and [**1**].) A power series $\sum a_n z^n$ has Ostrowski gaps if the set of indices of the zero coefficients has very long blocks.

THEOREM 2. *Let $g(z)=\sum a_n z^n$ have Ostrowski gaps, and be holomorphic in a connected open set Ω containing the origin. Then, there is a simply connected open set $\Omega^*\supset\Omega$ to which g can be extended.*

This is an application of the "two-constant" theorem which generalizes the Hadamard three circle theorem.

Theorems 1 and 2 together solve the Carlson problem. We use $h(\theta,f)$ for $\limsup r^{-1}\log|f(re^{i\theta})|$. Let $A\subset I$ and $UD(A)=1$, and suppose that f is entire of exponential type with $h(\pm\pi/2,f)\leqq C<\pi$, and $f(n)=0$ for $n\in A$. Choose S according to Theorem 1, and then an entire function ϕ of exponential type which vanishes on S but not identically. Since $D(S)=0$, we may require $h(\pm\pi/2,\phi)=0$. (See [**2**].) Put $F=f\phi$ and $g(z)=\sum F(n)z^n$. Since F vanishes on $A\cup S$ (which has very long blocks), g has Ostrowski gaps. Since $h(\pm\pi/2,F)=h(\pm\pi/2,f)<\pi$, the standard characterization theorem shows that g is holomorphic on the complement of a compact set K that omits the negative real axis, and vanishes at infinity. (See [**3**].) By Theorem 2, g can be extended to K and so vanishes everywhere, whence $F\equiv 0$ and $f\equiv 0$.

Theorem 1 seems to be confined to integers; a more subtle approach may be necessary to deal with more general zero sets [**9**].

REFERENCES

1. L. Bieberbach, *Analytische Fortsetzung*, Ergebnisse der Math. und ihrer Grenzgebiete, Heft 3, Springer-Verlag, Berlin, 1955. MR **16**, 913.

2. R. C. Buck, *An extension of Carlson's theorem*, Duke Math. J. **13** (1946), 345–349. MR **8**, 323.

3. ———, *A class of entire functions*, Duke Math. J. **13** (1946), 541–559. MR **8**, 371.

4. R. C. Buck and L. A. Rubel, *Some remarks on Carlson's theorem*, OOR Tech. Rpt. #3 (DA11022ORD875), 1954.

5. G. H. Hardy, *On two theorems of F. Carlson and S. Wigert*, Acta Math. **42** (1920), 327–339.

6. A. Ostrowski, *On representation of analytic functions by power series*, J. London Math. Soc. **1** (1926), 251–263.

7. L. A. Rubel, *Uniqueness properties of sets with blocks*, Proc. Amer. Math. Soc. **7** (1956), 1023–1026. MR **18**, 471.

8. ———, *Necessary and sufficient conditions for Carlson's theorem on entire functions*, Trans. Amer. Math. Soc. **83** (1956), 417–429; Also, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 601–603. MR **17**, 356, 1437; MR **18**, 471.

9. Paul Malliavin and L. A. Rubel, *On small entire functions of exponential type with given zeros*, Bull. Soc. Math. France **89** (1961), 175–206. MR **25** #5175; 26, 1543.

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