THE FIRST INITIAL-Boundary VALUE PROBLEM FOR SOME NONLINEAR TIME DEGENERATE PARABOLIC EQUATIONS

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Abstract. Consider the nonuniformly parabolic operator

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} - c(x,t,u)u_t + d(x,t)u,$$

where $u$, $a^{ij}$, $b^i$, $c$, $d$ are bounded, real-valued functions defined on a domain $D = \Omega \times [0, T] \subset \mathbb{R}^{n+1}$. Assume that $c(x,t,u)$ is Lipschitz continuous in $|\cdot|^2$ of $C_{0}(D)$, and that $c(x,t,u) \geq 0$ on $D$. Sufficient conditions on $c$ are found which guarantee existence of a unique solution $u \in C^{2,\alpha}$ to the first initial-boundary value problem $Lu = f(x,t)$, $u = \gamma$ on the normal boundary of $D$, where $\gamma \in C^{1,\alpha}$. Existence is proved by direct application of a fixed point theorem due to Schauder using existence of a solution to the linear problem as well as a priori estimates.

1. Introduction. We shall be concerned with existence of a solution to the first initial-boundary value problem for the second-order nonlinear nonuniformly parabolic operator

$$Lu = \sum_{i,j=1}^{n} a^{ij}u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} - c(x,t,u)u_t + d(x,t)u,$$

where $u$ and all coefficients of $L$ are real-valued functions defined for $(x,t) = (x_1, \cdots, x_n, t)$ in an $(n+1)$-dimensional, bounded, convex domain $D$. Subscripts will be used to denote differentiation.

We will assume that $L$ is parabolic; that is,

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \geq \gamma |\xi|^2 > 0$$

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487
for some $\gamma > 0$ and for any real vector $\xi \neq 0$. Assume that the coefficient $c(x, t, u) \geq 0$ but is not necessarily bounded away from zero. Since $c(x, t, u)$ may be zero for some $(x, t) \in D$, $L$ may be a degenerate parabolic operator. Note that $c$ may be a function of $u$ as well as of $x$ and $t$.

The author has already proved the existence of a unique solution to the first initial-boundary value problem for the linear equation

$$(1.3) \quad Mu = \sum_{i,j=1}^{n} a^{ij} u_{x_{i}x_{j}} + \sum_{i=1}^{n} b^{i} u_{x_{i}} - c(x, t)u_{t} + d(x, t)u = f(x, t)$$

in [4]. Existence of a solution to the nonlinear problem is proved with the aid of a fixed point theorem due to Schauder. We shall also use the fact that we can obtain a solution to the linear initial-boundary value problem together with a priori estimates obtained in [4].

This partial differential equation arises in applications to studies dealing with fluid flow through porous media. One such application [3] deals with diffusion and filtration of lipid-protein complexes and oxygen through the arterial wall and into surrounding tissue. Historically, the porosity coefficient $c$ has been assumed to be a positive constant, which is an average obtained in studying a healthy artery. However, tissue, as well as blood, is a heterogeneous composite of materials. In the case of a nonhealthy artery (as in atherosclerosis), it is precisely this heterogeneity which is of interest. The coefficient of porosity $c(x, t, u)$ is more accurately and usefully described as a coefficient of space, time, and pressure.

2. Notation and the linear first initial-boundary value problem. We shall use the same notation as A. Friedman [2, p. 40] to describe the domain $D$. As stated in the introduction, $D$ is a bounded, convex, $(n+1)$-dimensional domain in $R^{n+1}$, where $(x, t) = (x_1, \cdots, x_n, t)$ represents a variable point in $R^{n+1}$. $\partial D$ denotes the boundary of $D$. Thus, $\partial D = \bar{B} + B_T + S$, where $\bar{B}$ is a domain in $R^{n} \times \{0\}$, $B_T (T > 0)$ is a domain in $R^{n} \times \{T\}$, and $S$ is a manifold, not necessarily connected, in $R^{n} \times (0, T]$. $S + \bar{B}$ therefore denotes the parabolic, or normal, boundary of $D$.

Let $D_\tau = D \cap (R^{n} \times (0, \tau))$, $B_\tau = D \cap (R^{n} \times \{\tau\})$, and $S_\tau = S \cap (R^{n} \times (0, \tau))$. Assume that for each $\tau \in (0, T)$, $B_\tau$ is a domain. Then, for every $(x, \tau)$ in $D$, $0 < \tau < T$, if $S(x, \tau) = D_\tau + B_\tau$, then $(S(x, \tau))^- - S(x, \tau) = \bar{B} + S_\tau$, where $(S(x, \tau))^-$ denotes the closure of $(S(x, \tau))$. Also, assume that the following is a condition on $D$: There exists a simple continuous curve $\alpha$ connecting $B$ to $B_T$ along which the $t$-coordinate is nondecreasing.

Introduce the metric $d$ defined by

$$d(P, Q) = [\|x - \bar{x}\|^2 + |t - \bar{t}|]^{1/2}$$
where $P=(x, t)$, $Q=(\tilde{x}, \tilde{t})$, and $|x|=\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. Hölder continuity of a function $f$ is defined with respect to the metric $d$.

Suppose $\alpha \in (0, 1)$. Then let

$$|u|_{0}^{D} = \sup_{D} |u|, \quad H_{\alpha}^{D}(u) = \sup_{P, Q \in D} \frac{|u(P) - u(Q)|}{d(P, Q)^{\alpha}}, \quad |\tilde{u}|_{\alpha}^{D} = |u|_{0}^{D} + H_{\alpha}^{D}(u).$$

Then, $\tilde{C}_{\alpha}(D) = \{u: u: \overline{D} \rightarrow \mathbb{R}, |\tilde{u}|_{\alpha}^{D} < \infty\}$ is a Banach space with norm $|\cdot|_{\alpha}^{D}$; see [2, pp. 62–63].

Denote by $D_{x}^{m}$ any partial derivative of order $m$ with respect to the variables $x_{1}, \cdots, x_{n}$ and let $D_{t} = \partial / \partial t$. If $D_{x}u$, $D_{x}^{2}u$, $D_{t}u$ exist in $D$, then we define

$$|\tilde{u}|_{2+\alpha}^{D} = |\tilde{u}|_{0}^{D} + \sum |(D_{x}u)|_{\alpha}^{D} + \sum |(D_{x}^{2}u)|_{\alpha}^{D} + |(D_{t}u)|_{\alpha}^{D},$$

where the sums are taken over all partial derivatives of the indicated order. Let

$$\tilde{C}_{2+\alpha}(D) = \{u: u: \overline{D} \rightarrow \mathbb{R}, |\tilde{u}|_{2+\alpha}^{D} < \infty\}.$$

Then, $\tilde{C}_{2+\alpha}(D)$ is a Banach space with norm $|\cdot|_{2+\alpha}^{D}$; see [2]. When there is no confusion, we will drop the $D$ from $|\cdot|_{\alpha}^{D}$, $|\cdot|_{2+\alpha}^{D}$.

**Definition 2.1.** We say that $D$ has property $(E)$ if for every point $Q$ of $S$, there exists an $(n+1)$-dimensional neighborhood $V$ such that $V \cap S$ can be represented, for some $i$ $(1 \leq i \leq n)$, in the form

$$x_{i} = h(x_{1}, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n}, t),$$

where $h$, $D_{x}h$, $D_{x}^{2}h$, $D_{x}h$ are Hölder continuous of exponent $\alpha$.

**Definition 2.2.** If $D$ has property $(E)$ and if the functions $D_{x}D_{x}h$, $D_{x}^{2}h$ of the local representations of $S$ exist and are continuous functions, then we say $D$ has property $(E')$.

**Definition 2.3.** A function $\psi$ defined on $\overline{B} + S$ is said to belong to $\tilde{C}_{2+\alpha}(D)$ if there exist functions $\Psi$ in $\tilde{C}_{2+\alpha}(D)$ such that $\Psi = \psi$ on $\overline{B} + S$. Then $|\tilde{\psi}|_{2+\alpha}^{D}$ is defined by

$$|\tilde{\psi}|_{2+\alpha}^{D} = \inf_{\Psi} |\tilde{\Psi}|_{2+\alpha}^{D},$$

where the infimum is taken over all $\Psi \in \tilde{C}_{2+\alpha}(D)$ which coincide with $\psi$ on $\overline{B} + S$.

The following notation is one of convenience. If $u$ is a bounded real-valued function on a subset $S$ of $\mathbb{R}^{n+1}$, define

$$M(u; S) = \sup \{u(x, t): (x, t) \in S\}$$

and

$$m(u; S) = \inf \{u(x, t): (x, t) \in S\}.$$
In [4], the author solves the linear first initial-boundary value problem

\begin{equation}
Nu = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i(x, t) u_{x_i} - c(x, t) u_t + d(x, t) u = f(x, t) \quad \text{on } D + B_T, \\
u = \varphi \quad \text{on } \bar{B} + S
\end{equation}

(2.1)

without assuming that \( c \) has a positive minimum in \( D \). We now state this theorem precisely.

**Theorem 2.4.** Assume that \( a_{ij} \) is constant for each \( i, j \), that all coefficients of \( N \), defined in (2.1) are of class \( C^{1.1}(D) \), \( u \in C(D) \), \( u \in C^3(D) \), and \( a^{1.2} + b^1 \lambda \geq 1 \) for some \( \lambda > 0 \). Suppose, further, that the coefficients of \( N \) are uniformly Hölder continuous (exponent \( \alpha \)) in \( D \), \( |(a_{ij})^\alpha| \leq K_1 \), \( |(b^i)^\alpha| \leq K_1 \), \( |c^\alpha| \leq K_1 \), \( |\bar{d}|^\alpha \leq K_1 \), that \( m(c; \bar{B} + S) \geq \mu > 0 \) while \( m(c; D) \geq 0 \), that \( |f| < \infty \), and that (1.2) holds. If \( D \) has property \((E')\), \( \psi \in \mathcal{C}_{2+\alpha}(D) \), and \( N\psi = f \) on \( \partial B \), then there exists a unique solution \( u \) of the first initial-boundary value problem (2.1) and, furthermore, \( u \in \mathcal{C}_{2+\alpha}(D) \).

3. A priori estimates. Our proof for the nonlinear case requires the use of an a priori estimate for a solution to the linear problem. The following theorem was proved in [4].

**Theorem 3.1.** Suppose that the conditions of Theorem 2.4 hold, together with the added restriction that \( m(c; D) \geq \mu > 0 \). Then there exists a constant \( K \) depending only on \( K_1, K_2, \alpha, \) and \( D \) such that if \( u \) is a solution to \( Nu = f \), with \( u = \psi \) on \( \bar{B} + S \), and if \( u \in \mathcal{C}_{2+\alpha}(D) \), then

\begin{equation}
|\bar{u}|_{2+\alpha} \leq \mu^{-1/2} K (|\bar{\psi}|_{2+\alpha} + |f|_\alpha).
\end{equation}

The technique which was used in obtaining the solution in Theorem 2.4 involved perturbing the coefficient \( c \) by \( 1/k \) and considering the problem

\begin{equation}
N^k u = Nu - (1/k) u_t = f \quad \text{on } D + B_T, \\
u = \psi \quad \text{on } \bar{B} + S.
\end{equation}

(3.2)

By Friedman’s work, we were guaranteed a unique solution \( u_k \in \mathcal{C}_{2+\alpha}(D) \). We were able to show that the sequence \( \{u_k\} \) obtained in this manner is Cauchy in the Banach space \( \mathcal{C}_{2+\alpha}(D) \) and does, in fact, converge to the unique solution of the first initial-boundary value problem (2.1).

These \( u_k \), incidentally, satisfy

\begin{equation}
|(u_k)|_{2+\alpha} \leq k^{1/2} K (|\bar{\psi}|_{2+\alpha} + |f|_\alpha).
\end{equation}

(3.3)
If \(|\bar{v}|_{2+a} + |f|_a > 0\), there is some \(u_K\) such that \(|(u-u_K)|_{2+a} < |\bar{v}|_{2+a} + |f|_a\). Since \(u_K\) satisfies (3.1), we may conclude that
\[
|\bar{u}|_{2+a} = |(u - u_K + u_K)|_{2+a} 
\leq |(u - u_K)|_{2+a} + |(u_K)|_{2+a} 
\leq (|\bar{v}|_{2+a} + |f|_a) + K^{1/2}\bar{R}(|\bar{v}|_{2+a} + |f|_a) 
= (1 + K^{1/2}\bar{R})(|\bar{v}|_{2+a} + |f|_a).
\]

We have just proved the following theorem.

**Theorem 3.2.** Assume all the conditions of Theorem 2.4. Then the solution \(u\) of the first initial-boundary value problem (2.1) satisfies
\[
|u|_{2+a} \leq \bar{M}(|\bar{v}|_{2+a} + |f|_a),
\]
for some constant \(\bar{M}\) which is independent of \(m(c, D)\), provided \(|\bar{v}|_{2+a} + |f|_a > 0\).

4. The nonlinear problem. We now proceed to solve the first initial-boundary value problem (\(L\) defined as in (1.1)):
\[
Lu = f(x, t), \quad \text{on } D + B_T, \\
-\frac{\partial u}{\partial t} = \psi, \quad \text{on } B + S.
\]
For \(v\) a fixed element of \(C_{2+a}(D)\), consider the linear problem
\[
L_vu = \sum_{i,j=1}^n a^{ij}u_{x_i x_j} + \sum_{i=1}^n b^iu_{x_i} - c(x, t, v)u_t + du \\
= f(x, t), \quad \text{on } D + B_T, \\
u = \psi, \quad \text{on } B + S.
\]
Assuming that as a function of \(x\) and \(t\), \(c(x, t, v)\) satisfies the hypotheses of Theorem 2.4, we obtain a solution \(u \in C_{2+a}(D)\) to the initial-boundary value problem (4.2). \(u\) obviously depends upon \(v\). Hence, we define \(\phi: C_{2+a}(D) \to C_{2+a}(D)\) by \(u = \phi(v)\) is the unique solution to the initial-boundary value problem (4.2).

Under appropriate conditions on \(c\), we will show that
(i) \(\phi: A \to A\), where \(A\) is some closed convex subset of \(C_{2+a}(D)\); (ii) \(\phi\) is continuous in \(|\cdot|_{2+a}\) on \(A\).

We will then be able to apply the Schauder fixed point theorem of [2, p. 189] to obtain an element \(u \in C_{2+a}(D)\) such that \(\phi(u) = u\). Then \(u\) will be a solution to (4.1) and we will have proved the desired existence theorem.

An argument similar to that given by Friedman in [2] allows us to assume that \(\psi \equiv 0\) on \(B + S\). We will assume that \(c(x, t, u)\) is Lipschitz continuous with respect to \(u\) in the \(|\cdot|_a\) norm. That is, if \((x, t) \in D\)
and \( u_1, u_2 \in C_a(D) \), then
\[
|c(x, t, u_1) - c(x, t, u_2)|_a \leq \eta |u_1 - u_2|_a^D
\]
for some positive constant \( \eta \).

Let \( A = \{ u : |u|^2_{2,+,T} \leq 2M |f|_a^D, \ u = 0 \) on \( \bar{B} + S \} \) where \( f \) is the forcing function in (4.2) and \( M \) is the constant obtained in Theorem 3.2. \( A \) is clearly a closed convex subset of \( \mathcal{C}_{2,+,T}(D) \).

Now suppose that \( v \in A \) and \( h \in [0, 1] \). Then \( \phi(hv) \) satisfies \( L_h \phi(hv) = f, \ \phi(hv) = 0 \) on \( \bar{B} + S \). We note that \( L_h \phi(hv) = f \) can be rewritten in the form
\[
(4.3) \quad L_h \phi(v) = L_h \phi(v) - L_h \phi(hv) + f = F(v).
\]
Hence, \( \phi(v) \) satisfies:
\[
(4.4) \quad L_h \phi(v) = F(v) \quad \text{in} \ D + B_T, \\
\phi(v) = 0 \quad \text{on} \ \bar{B} + S.
\]
Using (4.3), we see that
\[
|(F(v))_a - f|_a^D \leq |(L_h \phi(v) - L_h \phi(hv))_a - f|_a^D \\
\leq K |(\phi(v) - \phi(hv))_a - f|_a^D \\
+ |(c(x, t, v) - c(x, t, hv))_a - f|_a^D + |(\phi(hv))_a - f|_a^D.
\]
But, \( |(\phi(hv))_a - f|_a^D \leq K_1 \) for some \( K_1 > 0 \) and for all \( h \in [0, 1] \). This follows from the proof of Theorem 6.4 of [4, pp. 53-54]. \( \phi(v) - \phi(hv) \) satisfies
\[
(4.5) \quad L_h(u) = [c(x, t, v) - c(x, t, hv)][\phi(hv)]_a \quad \text{in} \ D + B_T, \\
u = 0 \quad \text{on} \ \bar{B} + S.
\]
Applying Theorem 3.2 to the system (4.5), we have the estimate
\[
|(\phi(v) - \phi(hv))_a - f|_a^D \leq \bar{M} |(c(x, t, v) - c(x, t, hv))_a - f|_a^D.
\]
Therefore,
\[
|(F(v))_a - f|_a^D \leq K |(c(x, t, v) - c(x, t, hv))_a - f|_a^D \\
+ |(c(x, t, v) - c(x, t, hv))_a - f|_a^D + |(\phi(hv))_a - f|_a^D \\
\leq (K + 1) \eta |(v - hv)_a - f|_a^D K_1 + |f|_a^D \\
\leq (K + 1) \eta K_1 |1 - h| \|v\|_a^D + |f|_a^D.
\]
We have thus proved the inequality
\[
(4.6) \quad |(F(v))_a - f|_a^D \leq (K + 1) \eta K_1 |1 - h| \|v\|_a^D + |f|_a^D.
\]
Applying Theorem 3.2 to the system (4.4), we have the estimate

\[(\phi(v) - \phi(v_0))^{1}_{0} \leq M (F(v))^{1}_{0} \]

Inequalities (4.6) and (4.7) yield

\[
|\phi(v)| = [(\kappa + 1) \eta K_1 |1 - h| |\phi|^1_{0} + |\phi|^1_{0}] M.
\]

But \( v \in A \) implies \( |\phi|^1_{0} \leq 2M|f|^1_{0} \). Choose \( h \) so that \( 2(K + 1) \eta K_1 |1 - h| M \leq 1 \). Thus, \( |\phi(v)|^{1}_{0} \leq 2M|f|^1_{0} \). We conclude that \( \phi \) maps \( A \) to \( A \).

It is easy to show that \( \phi \) is continuous with respect to \( \| \cdot \|_A \) in \( A \).

Note that if \( v_0 \) and \( v \) are in \( A \), \( \phi(v_0) - \phi(v) \) satisfies

\[
L_{v_0}(u) = [c(x, t, v_0) - c(x, t, v)](\phi(v)) \text{ in } D + B_T,
\]

\[u = 0 \text{ on } B + S.
\]

Applying Theorem 3.2 to the system (3.8) we obtain the estimate

\[
|\phi(v) - \phi(v_0)|^{1}_{0} \leq M |(c(x, t, v_0) - c(x, t, v)) - (\phi(v))|^{1}_{0} \]

Recall that \( |(\phi(v))|^{1}_{0} \leq K_1 \). This uniform bound, together with Lipschitz continuity of \( c(x, t, u) \) with respect to \( u \), and (4.9) give

\[
|\phi(v) - \phi(v_0)|^{1}_{0} \leq M K_1 |v_0 - v|^{1}_{0}.
\]

Thus, \( \phi \) is continuous with respect to \( | \cdot |_A \) in \( A \).

A fixed point theorem due to Schauder [2, p. 189] can be used to conclude that there exists a function \( u \in A \) for which \( \phi(u) = u \). We have thus proved the following theorem.

**Theorem 4.1** Assume that the conditions of Theorem 2.4 hold for the operator \( L \), defined in (1.1), and suppose also that \( c(x, t, u) \) is Lipschitz continuous with respect to \( u \) in the \( | \cdot |_A \) norm. If \( D \) has property (E'), \( \psi \in C_{2+\alpha}(D) \), and \( L\psi = f \) on \( \partial B \), then there exists a solution \( u \), of the nonlinear first initial-boundary value problem (4.1) and furthermore, \( u \in C_{2+\alpha}(D) \).

**Theorem 4.2.** The solution \( u \) obtained in Theorem 4.1 is unique.

**Proof.** The uniqueness of the solution to the nonlinear problem (4.1) follows from uniqueness of the solution to the linear problem of Theorem 2.4 by a well-known argument using the mean-value theorem. See for example a paper by A. P. Calderón [1, p. 35], or extend in a straightforward manner the argument given by A. Friedman [2, p. 41].
References


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