

THE FIRST INITIAL-BOUNDARY VALUE PROBLEM
 FOR SOME NONLINEAR TIME DEGENERATE
 PARABOLIC EQUATIONS

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ABSTRACT. Consider the nonuniformly parabolic operator

$$Lu = \sum_{i,j=1}^n a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} - c(x,t,u)u_t + d(x,t)u,$$

where u, a^{ij}, b^i, c, d are bounded, real-valued functions defined on a domain $D = \Omega \times [0, T] \subset \mathbb{R}^{n+1}$. Assume that $c(x, t, u)$ is Lipschitz continuous in $|\cdot|_\alpha^2$ of $C_\alpha(D)$, and that $c(x, t, u) \geq 0$ on D . Sufficient conditions on c are found which guarantee existence of a unique solution $u \in \bar{C}_{2+\alpha}$ to the first initial-boundary value problem $Lu = f(x, t), u = \psi$, on the normal boundary of D , where $\psi \in \bar{C}_{2+\alpha}$. Existence is proved by direct application of a fixed point theorem due to Schauder using existence of a solution to the linear problem as well as a priori estimates.

1. **Introduction.** We shall be concerned with existence of a solution to the first initial-boundary value problem for the second-order nonlinear nonuniformly parabolic operator

$$(1.1) \quad Lu = \sum_{i,j=1}^n a^{ij}u_{x_i x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} - c(x,t,u)u_t + d(x,t)u,$$

where u and all coefficients of L are real-valued functions defined for $(x, t) = (x_1, \dots, x_n, t)$ in an $(n+1)$ -dimensional, bounded, convex domain D . Subscripts will be used to denote differentiation.

We will assume that L is parabolic; that is,

$$(1.2) \quad \sum_{i,j=1}^n a^{ij}\xi_i\xi_j \geq \gamma|\xi|^2 > 0$$

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for some $\gamma > 0$ and for any real vector $\xi \neq 0$. Assume that the coefficient $c(x, t, u) \geq 0$ but is not necessarily bounded away from zero. Since $c(x, t, u)$ may be zero for some $(x, t) \in D$, L may be a degenerate parabolic operator. Note that c may be a function of u as well as of x and t .

The author has already proved the existence of a unique solution to the first initial-boundary value problem for the linear equation

$$(1.3) \quad Mu = \sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} - c(x, t) u_t + d(x, t) u = f(x, t)$$

in [4]. Existence of a solution to the nonlinear problem is proved with the aid of a fixed point theorem due to Schauder. We shall also use the fact that we can obtain a solution to the linear initial-boundary value problem together with a priori estimates obtained in [4].

This partial differential equation arises in applications to studies dealing with fluid flow through porous media. One such application [3] deals with diffusion and filtration of lipid-protein complexes and oxygen through the arterial wall and into surrounding tissue. Historically, the porosity coefficient c has been assumed to be a positive constant, which is an average obtained in studying a healthy artery. However, tissue, as well as blood, is a heterogeneous composite of materials. In the case of a nonhealthy artery (as in atherosclerosis), it is precisely this heterogeneity which is of interest. The coefficient of porosity $c(x, t, u)$ is more accurately and usefully described as a coefficient of space, time, and pressure.

2. Notation and the linear first initial-boundary value problem. We shall use the same notation as A. Friedman [2, p. 40] to describe the domain D . As stated in the introduction, D is a bounded, convex, $(n+1)$ -dimensional domain in R^{n+1} , where $(x, t) = (x_1, \dots, x_n, t)$ represents a variable point in R^{n+1} . ∂D denotes the boundary of D . Thus, $\partial D = \bar{B} + B_T + S$, where B is a domain in $R^n \times \{0\}$, B_T ($T > 0$) is a domain in $R^n \times \{T\}$, and S is a manifold, not necessarily connected, in $R^n \times (0, T)$. $S + \bar{B}$ therefore denotes the parabolic, or normal, boundary of D .

Let $D_\tau = D \cap (R^n \times (0, \tau))$, $B_\tau = D \cap (R^n \times \{\tau\})$, and $S_\tau = S \cap (R^n \times (0, \tau))$. Assume that for each $\tau \in (0, T)$, B_τ is a domain. Then, for every (x, τ) in D , $0 < \tau < T$, if $S(x, \tau) = D_\tau + B_\tau$, then $(S(x, \tau))^- - S(x, \tau) = \bar{B} + S_\tau$, where $(S(x, \tau))^-$ denotes the closure of $(S(x, \tau))$. Also, assume that the following is a condition on D : There exists a simple continuous curve α connecting B to B_T along which the t -coordinate is nondecreasing.

Introduce the metric d defined by

$$d(P, Q) = [|x - \bar{x}|^2 + |t - \bar{t}|^2]^{1/2}$$

where $P=(x, t)$, $Q=(\bar{x}, \bar{t})$, and $|x|=(\sum_{i=1}^n x_i^2)^{1/2}$. Hölder continuity of a function f is defined with respect to the metric d .

Suppose $\alpha \in (0, 1)$. Then let

$$|u|_0^D = \sup_D |u|, \quad \bar{H}_\alpha^D(u) = \sup_{P, Q \in D} \frac{|u(P) - u(Q)|}{d(P, Q)^\alpha}, \quad |\bar{u}|_\alpha^D = |u|_0^D + \bar{H}_\alpha^D(u).$$

Then, $\bar{C}_\alpha(D) = \{u: u: \bar{D} \rightarrow R, |\bar{u}|_\alpha^D < \infty\}$ is a Banach space with norm $|\bar{\cdot}|_\alpha^D$; see [2, pp. 62-63].

Denote by D_x^m any partial derivative of order m with respect to the variables x_1, \dots, x_n and let $D_t = \partial/\partial t$. If $D_x u, D_x^2 u, D_t u$ exist in D , then we define

$$|\bar{u}|_{2+\alpha}^D = |\bar{u}|_\alpha^D + \sum |(D_x u)^{-}|_\alpha^D + \sum |(D_x^2 u)^{-}|_\alpha^D + |(D_t u)^{-}|_\alpha^D,$$

where the sums are taken over all partial derivatives of the indicated order. Let

$$\bar{C}_{2+\alpha}(D) = \{u: u: \bar{D} \rightarrow R, |\bar{u}|_{2+\alpha}^D < \infty\}.$$

Then, $\bar{C}_{2+\alpha}(D)$ is a Banach space with norm $|\bar{\cdot}|_{2+\alpha}^D$; see [2]. When there is no confusion, we will drop the D from $|\bar{\cdot}|_\alpha^D, |\bar{\cdot}|_{2+\alpha}^D$.

DEFINITION 2.1. We say that D has property (\bar{E}) if for every point Q of \bar{S} , there exists an $(n+1)$ -dimensional neighborhood V such that $V \cap \bar{S}$ can be represented, for some i ($1 \leq i \leq n$), in the form

$$x_i = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t),$$

where $h, D_x h, D_x^2 h, D_t h$ are Hölder continuous of exponent α .

DEFINITION 2.2. If D has property (\bar{E}) and if the functions $D_x D_t h, D_x^2 h$ of the local representations of \bar{S} exist and are continuous functions, then we say D has property (E') .

DEFINITION 2.3. A function ψ defined on $\bar{B}+S$ is said to belong to $\bar{C}_{2+\alpha}(D)$ if there exist functions Ψ in $\bar{C}_{2+\alpha}(D)$ such that $\Psi = \psi$ on $\bar{B}+S$. Then $|\bar{\psi}|_{2+\alpha}^D$ is defined by

$$|\bar{\psi}|_{2+\alpha}^D = \inf_{\Psi} |\bar{\Psi}|_{2+\alpha}^D,$$

where the infimum is taken over all $\Psi \in \bar{C}_{2+\alpha}(D)$ which coincide with ψ on $\bar{B}+S$.

The following notation is one of convenience. If u is a bounded real-valued function on a subset S of R^{n+1} , define

$$M(u; S) = \sup\{u(x, t): (x, t) \in S\}$$

and

$$m(u; S) = \inf\{u(x, t): (x, t) \in S\}.$$

In [4], the author solves the linear first initial-boundary value problem

$$\begin{aligned}
 (2.1) \quad Nu &= \sum_{i,j=1}^n a^{ij}u_{x_i x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} - c(x,t)u_t + d(x,t)u \\
 &= f(x,t) \quad \text{on } D + B_T, \\
 u &= \psi \quad \text{on } \bar{B} + S
 \end{aligned}$$

without assuming that c has a positive minimum in D . We now state this theorem precisely.

THEOREM 2.4. *Assume that a^{ij} is constant for each i, j , that all coefficients of N , defined in (2.1) are of class $C^{1,1}(\bar{D})$, $u \in C(\bar{D})$, $u \in C^3(D)$, and $a^{11}\lambda^2 + b^1\lambda \geq 1$ for some $\lambda > 0$. Suppose, further, that the coefficients of N are uniformly Hölder continuous (exponent α) in D , $|(a^{ij})^-|_\alpha \leq K_1$, $|(b^i)^-|_\alpha \leq K_1$, $|\bar{c}|_\alpha \leq K_1$, $|\bar{d}|_\alpha \leq K_1$, that $m(c; \bar{B} + S) \geq \mu > 0$ while $m(c; D) \geq 0$, that $|f|_\alpha < \infty$, and that (1.2) holds. If D has property (E'), $\psi \in \bar{C}_{2+\alpha}(D)$, and $N\psi = f$ on ∂B , then there exists a unique solution u of the first initial-boundary value problem (2.1) and, furthermore, $u \in \bar{C}_{2+\alpha}(D)$.*

3. A priori estimates. Our proof for the nonlinear case requires the use of an a priori estimate for a solution to the linear problem. The following theorem was proved in [4].

THEOREM 3.1. *Suppose that the conditions of Theorem 2.4 hold, together with the added restriction that $m(c; D) \geq \mu > 0$. Then there exists a constant K depending only on K_1, K_2, α , and D such that if u is a solution to $Nu = f$, with $u = \psi$ on $\bar{B} + S$, and if $u \in \bar{C}_{2+\alpha}(D)$, then*

$$(3.1) \quad |\bar{u}|_{2+\alpha} \leq \mu^{-1/2}K(|\bar{\psi}|_{2+\alpha} + |f|_\alpha).$$

The technique which was used in obtaining the solution in Theorem 2.4 involved perturbing the coefficient c by $1/k$ and considering the problem

$$\begin{aligned}
 (3.2) \quad N^k u &= Nu - (1/k)u_t = f \quad \text{on } D + B_T, \\
 u &= \psi \quad \text{on } \bar{B} + S.
 \end{aligned}$$

By Friedman's work, we were guaranteed a unique solution $u_k \in \bar{C}_{2+\alpha}(D)$. We were able to show that the sequence $\{u_k\}$ obtained in this manner is Cauchy in the Banach space $\bar{C}_{2+\alpha}(D)$ and does, in fact, converge to the unique solution of the first initial-boundary value problem (2.1).

These u_k , incidentally, satisfy

$$(3.3) \quad |(u_k)^-|_{2+\alpha} \leq k^{1/2}K(|\bar{\psi}|_{2+\alpha} + |f|_\alpha).$$

If $|\bar{\psi}|_{2+\alpha} + |f|_\alpha > 0$, there is some u_K such that $|(u - u_K)^-|_{2+\alpha} < |\bar{\psi}|_{2+\alpha} + |f|_\alpha$. Since u_K satisfies (3.1), we may conclude that

$$\begin{aligned} |\bar{u}|_{2+\alpha} &= |(u - u_K + u_K)^-|_{2+\alpha} \leq |(u - u_K)^-|_{2+\alpha} + |(u_K)^-|_{2+\alpha} \\ &\leq (|\bar{\psi}|_{2+\alpha} + |f|_\alpha) + K^{1/2}\bar{K}(|\bar{\psi}|_{2+\alpha} + |f|_\alpha) \\ &= (1 + K^{1/2}\bar{K})(|\bar{\psi}|_{2+\alpha} + |f|_\alpha). \end{aligned}$$

We have just proved the following theorem.

THEOREM 3.2. *Assume all the conditions of Theorem 2.4. Then the solution u of the first initial-boundary value problem (2.1) satisfies*

$$(3.4) \quad |\bar{u}|_{2+\alpha} \leq \bar{M}(|\bar{\psi}|_{2+\alpha} + |f|_\alpha),$$

for some constant \bar{M} which is independent of $m(c; D)$, provided $|\bar{\psi}|_{2+\alpha} + |f|_\alpha > 0$.

4. The nonlinear problem. We now proceed to solve the first initial-boundary value problem (L defined as in (1.1)):

$$(4.1) \quad \begin{aligned} Lu &= f(x, t), \quad \text{on } D + B_T, \\ u &= \psi, \quad \text{on } \bar{B} + S. \end{aligned}$$

For v a fixed element of $\bar{C}_{2+\alpha}(D)$, consider the linear problem

$$(4.2) \quad \begin{aligned} L_v u &= \sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} - c(x, t, v)u_t + du \\ &= f(x, t), \quad \text{on } D + B_T, \\ u &= \psi, \quad \text{on } \bar{B} + S. \end{aligned}$$

Assuming that as a function of x and t , $c(x, t, v)$ satisfies the hypotheses of Theorem 2.4, we obtain a solution $u \in \bar{C}_{2+\alpha}(D)$ to the initial-boundary value problem (4.2). u obviously depends upon v . Hence, we define $\phi: \bar{C}_{2+\alpha}(D) \rightarrow \bar{C}_{2+\alpha}(D)$ by $u = \phi(v)$ is the unique solution to the initial-boundary value problem (4.2).

Under appropriate conditions on c , we will show that

- (i) $\phi: A \rightarrow A$, where A is some closed convex subset of $\bar{C}_{2+\alpha}(D)$;
- (ii) ϕ is continuous in $|\cdot|_{2+\alpha}^D$ on A .

We will then be able to apply the Schauder fixed point theorem of [2, p. 189] to obtain an element $u \in \bar{C}_{2+\alpha}(D)$ such that $\phi(u) = u$. Then u will be a solution to (4.1) and we will have proved the desired existence theorem.

An argument similar to that given by Friedman in [2] allows us to assume that $\psi \equiv 0$ on $\bar{B} + S$. We will assume that $c(x, t, u)$ is Lipschitz continuous with respect to u in the $|\cdot|_\alpha$ norm. That is, if $(x, t) \in D$

and $u_1, u_2 \in C_x(D)$, then

$$|(c(x, t, u_1) - c(x, t, u_2))^-|_\alpha^D \leq \eta |(u_1 - u_2)^-|_\alpha^D$$

for some positive constant η .

Let $A = \{u : |\bar{u}|_{2+\alpha}^D \leq 2\bar{M}|f|_\alpha^D, u=0 \text{ on } \bar{B}+S\}$ where f is the forcing function in (4.2) and \bar{M} is the constant obtained in Theorem 3.2. A is clearly a closed convex subset of $\bar{C}_{2+\alpha}(D)$.

Now suppose that $v \in A$ and $h \in [0, 1]$. Then $\phi(hv)$ satisfies $L_{hv}\phi(hv)=f, \phi(hv)=0$ on $\bar{B}+S$. We note that $L_{hv}\phi(hv)=f$ can be rewritten in the form

$$(4.3) \quad L_v\phi(v) = L_v\phi(v) - L_{hv}\phi(hv) + f \equiv F(v).$$

Hence, $\phi(v)$ satisfies:

$$(4.4) \quad \begin{aligned} L_v\phi(v) &= F(v) && \text{in } D + B_T, \\ \phi(v) &= 0 && \text{on } \bar{B} + S. \end{aligned}$$

Using (4.3), we see that

$$\begin{aligned} |(F(v))^-|_\alpha^D &\leq |(L_v\phi(v) - L_{hv}\phi(hv))^-|_\alpha^D + |f|_\alpha^D \\ &\leq K |(\phi(v) - \phi(hv))^-|_{2+\alpha}^D \\ &\quad + |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D |([\phi(hv)]_t)^-|_\alpha^D + |f|_\alpha^D. \end{aligned}$$

But, $|([\phi(hv)]_t)^-|_\alpha^D \leq K_1$ for some $K_1 > 0$ and for all $h \in [0, 1]$. This follows from the proof of Theorem 6.4 of [4, pp. 53-54]. $\phi(v) - \phi(hv)$ satisfies

$$(4.5) \quad \begin{aligned} L_v(u) &= [c(x, t, v) - c(x, t, hv)][\phi(hv)]_t && \text{in } D + B_T, \\ u &= 0 && \text{on } \bar{B} + S. \end{aligned}$$

Applying Theorem 3.2 to the system (4.5), we have the estimate

$$|(\phi(v) - \phi(hv))^-|_{2+\alpha}^D \leq \bar{M} |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D |([\phi(hv)]_t)^-|_\alpha^D.$$

Therefore,

$$\begin{aligned} |(F(v))^-|_\alpha^D &\leq K |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D |([\phi(hv)]_t)^-|_\alpha^D \\ &\quad + |(c(x, t, v) - c(x, t, hv))^-|_\alpha^D |([\phi(hv)]_t)^-|_\alpha^D + |f|_\alpha^D \\ &\leq (K + 1)\eta |(v - hv)^-|_\alpha^D K_1 + |f|_\alpha^D \\ &\leq (K + 1)\eta K_1 |1 - h| |\bar{v}|_\alpha^D + |f|_\alpha^D. \end{aligned}$$

We have thus proved the inequality

$$(4.6) \quad |(F(v))^-|_\alpha^D \leq (K + 1)\eta K_1 |1 - h| |\bar{v}|_\alpha^D + |f|_\alpha^D.$$

Applying Theorem 3.2 to the system (4.4), we have the estimate

$$(4.7) \quad |(\phi(v))^-|_{2+\alpha}^D \leq \bar{M} |(F(v))^-|_{\alpha}^D.$$

Inequalities (4.6) and (4.7) yield

$$|(\phi(v))^-|_{2+\alpha}^D \leq [(K+1)\eta K_1 |1-h| |\bar{v}|_{\alpha}^D + |\bar{f}|_{\alpha}^D] \bar{M}.$$

But $v \in A$ implies $|\bar{v}|_{\alpha}^D \leq |\bar{v}|_{2+\alpha}^D \leq 2\bar{M} |\bar{f}|_{\alpha}^D$. Choose h so that $2(K+1)\eta K_1 |1-h| \bar{M} \leq 1$. Thus, $|(\phi(v))^-|_{2+\alpha}^D \leq 2\bar{M} |\bar{f}|_{\alpha}^D$. We conclude that ϕ maps A to A .

It is easy to show that ϕ is continuous with respect to $|\bar{\cdot}|_{2+\alpha}^D$ in A . Note that if v_0 and v are in A , $\phi(v_0) - \phi(v)$ satisfies

$$(4.8) \quad \begin{aligned} L_{v_0}(u) &= [c(x, t, v_0) - c(x, t, v)] [\phi(v)]_t \quad \text{in } D + B_T, \\ u &= 0 \quad \text{on } \bar{B} + S. \end{aligned}$$

Applying Theorem 3.2 to the system (3.8) we obtain the estimate

$$(4.9) \quad |(\phi(v) - \phi(v_0))^-|_{2+\alpha}^D \leq \bar{M} |(c(x, t, v_0) - c(x, t, v))^-|_{\alpha}^D |([\phi(v)]_t)^-|_{\alpha}^D.$$

Recall that $|([\phi(v)]_t)^-|_{\alpha}^D \leq K_1$. This uniform bound, together with Lipschitz continuity of $c(x, t, u)$ with respect to u , and (4.9) give

$$(4.10) \quad |(\phi(v) - \phi(v_0))^-|_{2+\alpha}^D \leq \bar{M} K_1 \eta |(v_0 - v)^-|_{\alpha}^D.$$

Thus, ϕ is continuous with respect to $|\bar{\cdot}|_{2+\alpha}$ in A .

A fixed point theorem due to Schauder [2, p. 189] can be used to conclude that there exists a function $u \in A$ for which $\phi(u) = u$. We have thus proved the following theorem.

THEOREM 4.1 *Assume that the conditions of Theorem 2.4 hold for the operator L , defined in (1.1), and suppose also that $c(x, t, u)$ is Lipschitz continuous with respect to u in the $|\bar{\cdot}|_{\alpha}$ norm. If D has property (E'), $\psi \in \bar{C}_{2+\alpha}(D)$, and $L\psi = f$ on ∂B , then there exists a solution u , of the nonlinear first initial-boundary value problem (4.1) and furthermore, $u \in \bar{C}_{2+\alpha}(D)$.*

THEOREM 4.2. *The solution u obtained in Theorem 4.1 is unique.*

PROOF. The uniqueness of the solution to the nonlinear problem (4.1) follows from uniqueness of the solution to the linear problem of Theorem 2.4 by a well-known argument using the mean-value theorem. See for example a paper by A. P. Calderón [1, p. 35], or extend in a straightforward manner the argument given by A. Friedman [2, p. 41].

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