PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

MONOTONE AND COMONOTONE APPROXIMATION

E. PASSOW AND L. RAYMON

ABSTRACT. Jackson type theorems are obtained for monotone and comonotone approximation. Namely

(i) If $f(x)$ is a function such that the kth difference of f is $\geqq 0$ on $[a, b]$ then the degree of approximation of f by nth degree polynomials with kth derivative $\geqq 0$ is $0[\omega(f; 1/n^{1-\varepsilon})]$ for any $\varepsilon>0$, where $\omega(f; \delta)$ is the modulus of continuity of f on $[a, b]$.

(ii) If $f(x)$ is piecewise monotone on $[a, b]$ then the degree of approximation of f by nth degree polynomials comonotone with f is $0[\omega(f; 1/n^{1-\varepsilon})]$ for any $\varepsilon>0$.

The degree of approximation of a real function $f \in C[a, b]$ by a space of functions $\mathscr{P}$ is

$$E(f; \mathscr{P}) = \inf_{P \in \mathscr{P}} \|f - P\|,$$

where the norm is the ordinary sup norm. Jackson's classic theorem states that the degree of approximation of a function $f \in C[a, b]$ by the space $\mathscr{P}_n$ of algebraic polynomials of degree $\leqq n$ satisfies

$$E(f; \mathscr{P}_n) = E_n(f) \leqq C\omega(f; 1/n), \tag{1}$$

where $C>0$ is a constant not depending on n or f, and $\omega(f; \delta)$ is the modulus of continuity of f. It is natural to ask to what extent the degree of approximation to f is affected by replacing the space of approximating functions $\mathscr{P}_n$ by another (restricted) space $\mathscr{P}_n^* \subset \mathscr{P}_n$. In this article we address ourselves to two related questions of this type, (A) monotone approximation and (B) comonotone approximation.

(A) *Monotone approximation.* How closely can one approximate a monotone function f on $[a, b]$ by a polynomial that is monotone on $[a, b]$? I.e., what is the degree of approximation of f by the space of polynomials of degree $\leqq n$ that are monotone on $[a, b]$? More generally, if $\mathscr{P}_{n,k}$ denotes the space of polynomials P of degree $\leqq n$ satisfying $P^{(k)}(x) \geqq 0$ on $[a, b]$, then what is the degree of approximation $E(f; \mathscr{P}_{n,k}) = E_{n,k}(f)$ where $f(x)$ is a function whose kth difference $\Delta^k f$ is always $\geqq 0$ on $[a, b]$?

Received by the editors February 26, 1973.

AMS (MOS) subject classifications (1970). Primary 41A25, 41A10, 42A08.

Key words and phrases. Monotone approximation, comonotone approximation, piecewise monotone approximation, Jackson theorem.

These questions were first raised by Shisha in [5], where he proved that if $f^{(k)}(x) \geqq 0$ and $f^{(p)}(x) \in \text{Lip } 1$, where $1 \leqq k \leqq p$, then

$$(2) \qquad \begin{aligned} E_{n,k}(f) &\leqq C(\pi/4)^{p-k+1}(b-a)^{p+1}\left[k!\prod_{j=k}^{p}(n+1-j)\right]^{-1} \\ &\leqq \frac{C_{p,k}}{(n-p)^{p-k+1}}. \end{aligned}$$

Roulier [4] has obtained results that represent some improvement over Shisha's in certain cases where $k=p\geqq 2$. If f is not assumed to be in C^p for any $p\geqq 1$, the question of an estimate on the order of magnitude of $E_{n,k}$ remains. For the case $k=1$, Lorentz and Zeller [1] have obtained a very satisfying result. They have shown that for a monotone function f

$$E_{n,1}(f) = O[\omega(f;\, 1/n)].$$

This is the same order of magnitude as that given by Jackson's theorem for "unrestricted" approximation (1). In Theorem 3 we are able to show that if the kth difference $\Delta^k f$ of f is $\geqq 0$, then for every $\varepsilon > 0$

$$E_{n,k}(f) = o[\omega(f;\, 1/n^{1-\varepsilon})].$$

(B) *Comonotone approximation.* f will be called *piecewise monotone* if it has only a finite number of local maxima and minima in $[a, b]$. The local maxima and minima of f in $[a, b]$ together with the endpoints a, b will be referred to as the *peaks* of f. If g is nondecreasing on the subintervals of $[a, b]$ on which f is nondecreasing, and nonincreasing on those subintervals on which f is nonincreasing, then g is said to be *comonotone* with f. Given a piecewise monotone function $f(x)$ let $\mathscr{P}_n^*(f)$ denote the space of all polynomials of degree $\leqq n$ that are comonotone with f on $[a, b]$; let $E_n^*(f)$ denote the comonotone degree of approximation of f; i.e.,

$$E_n^*(f) = E_n[f;\, \mathscr{P}_n^*(f)].$$

By Jackson's theorem $E_n(f) = O[\omega(f;\, 1/n)]$. What is the order of magnitude of $E_n^*(f)$? Newman, Passow and Raymon [2] have obtained results of a modified nature. They have shown that for n sufficiently large there is $P \in \mathscr{P}_n$ such that $\|f-P\| < C\omega(f;\, 1/n)$ where f and P are comonotone except in certain neighborhoods (whose diameters tend to zero with n) of the peaks. Also, Passow and Raymon have obtained an estimate for "perfectly" comonotone approximation for functions in C^p [3]: If $f(x)$ has k peaks and $f \in C^p[a, b]$ with $p > k$ and with $f^{(p)} \in \text{Lip } 1\,[a, b]$, then

$$(3) \qquad E_n^*(f) \leqq (b-a)^{k+1}(C/n)^{p-k-1}$$

whenever $n > 2p$, where C is independent of n, p, f and k.

Let S be a set of functions. We shall use the following notation:

$$E_n^*(S) = \sup_{f \in S} E_n^*(f).$$

Theorem 1 relates the comonotone degree of approximation $E_n^*(f)$ of an arbitrary function f to $E_n^*(S)$ where S is a class of functions satisfying certain smoothness conditions. Theorem 2 is proved easily from (3) and Theorem 1.

THEOREM 1. *Let S^p denote the set of functions g in $C^p[a, b]$ such that $g^{(p)}$ is a contraction on $[a, b]$ (i.e., $\omega(g^{(p)}; \delta) \leqq \delta$ for all $\delta > 0$). Let $a = x_0 < x_1 < \cdots < x_k = b$ be the peaks of a piecewise monotone function $f(x)$ on $[a, b]$; let $\delta = \frac{1}{2} \min_{1 \leqq i \leqq k} |x_i - x_{i-1}|$. Let $\lambda = \lambda_n = [E_n^*(S^p)]^{1/(p+1)}$. Then*

$$E_n^*(f) \leqq p^2 2^{p+1} \omega(f; \lambda_n)$$

whenever $p\lambda_n < \delta$.

PROOF. Let $f^*(x)$ be defined on $[a, b + p\lambda]$ as follows:

$$\begin{aligned} f^*(x) &= f(x_i), \qquad x_i \leqq x \leqq x_i + p\lambda,\ i = 1, 2, \cdots, k, \\ &= f(x), \qquad \text{for all other } x. \end{aligned}$$

$f^*(x)$ is comonotone with $f(x)$ on $[a, b]$. In addition, the monotonicity of $f^*(x)$ on $[x_{i-1}, x_i]$ extends to $[x_{i-1}, x_i + p\lambda]$, $i = 1, 2, \cdots, k$. From the definition of $f^*(x)$ and by the sublinear property of the modulus of continuity we deduce

$$\omega(f^*; \lambda) \leqq \omega(f; p\lambda) \leqq p\omega(f; \lambda), \tag{4}$$

and

$$\|f - f^*\| \leqq \omega(f; p\lambda) \leqq p\omega(f; \lambda). \tag{5}$$

Let

$$g(x) = \frac{1}{\lambda^{p+1}} \int_x^{x+\lambda} \int_{t_p}^{t_p+\lambda} \int_{t_{p-1}}^{t_{p-1}+\lambda} \cdots \int_{t_1}^{t_1+\lambda} f^*(t)\, dt\, dt_1\, dt_2 \cdots dt_p.$$

We shall show that f and g are comonotone. If $f(x)$ is nondecreasing on $[x_{i-1}, x_i]$, then $f^*(x)$ is nondecreasing on $[x_{i-1}, x_i + p\lambda]$ and $g_1(x) = \int_x^{x+\lambda} f^*(t)\, dt$ is nondecreasing on $[x_{i-1}, x_i + (p-1)\lambda]$; $g_2(x) = \int_x^{x+\lambda} \int_{t_1}^{t_1+\lambda} f^*(t)\, dt\, dt_1$ is nondecreasing on $[x_{i-1}, x_i + (p-2)\lambda]$; Iterating the procedure p times, we conclude that $g(x)$ is nondecreasing on $[x_{i-1}, x_i]$. Similarly, if $f(x)$ is nonincreasing on $[x_{i-1}, x_i]$, then $g(x)$ is nonincreasing on the same interval, and $g(x)$ is comonotone with $f(x)$.

Applying the Fundamental Theorem of Calculus p times to $g(x)$ we conclude that

$$g^{(p)}(x) = \frac{1}{\lambda^{p+1}} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \int_{x+(p-j)\lambda}^{x+(p-j+1)\lambda} f^*(t)\, dt.$$

Then, since f^* is continuous except at a finite number of points,

$$|g^{(p+1)}(x)| \leqq \frac{2^p}{\lambda^{p+1}}\, \omega(f^*, \lambda) \leqq \frac{p2^p}{\lambda^{p+1}}\, \omega(f; \lambda), \quad \text{by (4).}$$

Hence $\lambda^{p+1}g(x)/p2^p\omega(f; \lambda) \in S^p$. Then there is some polynomial $Q(x) \in \mathscr{P}_n^*(f)$ such that

$$\|\lambda^{p+1}g(x)/p2^p\omega(f; \lambda) - Q(x)\| \leqq E_n^*(S^p) = \lambda^{p+1}.$$

Then, if $P(x)=p2^p\omega(f; \lambda)Q(x)/\lambda^{p+1}$, $P \in \mathscr{P}_n^*(f)$ and

$$\|g(x) - P(x)\| \leqq p2^p\omega(f; \lambda). \tag{6}$$

Also

$$\begin{aligned} \|g - f^*\| &= \left\| \frac{1}{\lambda^{p+1}} \int_x^{x+\lambda} \int_{t_p}^{t_p+\lambda} \cdots \int_{t_1}^{t_1+\lambda} [f^*(t) - f^*(x)]\, dt\, dt_1 \cdots dt_p \right\| \\ &\leqq \frac{\omega(f^*; p\lambda)}{\lambda^{p+1}} \left\| \int_x^{x+\lambda} \int_{t_p}^{t_p+\lambda} \cdots \int_{t_1}^{t_1+\lambda} dt\, dt_1 \cdots dt_p \right\| \\ &\leqq \omega(f; p^2\lambda) \leqq p^2\omega(f; \lambda). \end{aligned} \tag{7}$$

Now, from (5), (6) and (7), we have

$$\begin{aligned} E_n^*(f) &\leqq \|f - P\| \leqq \|f - f^*\| + \|f^* - g\| + \|g - P\| \\ &\leqq (p + p^2 + p2^p)\omega(f; \lambda) \leqq p^2 2^{p+1}\omega(f; \lambda), \end{aligned}$$

and the proof is complete.

THEOREM 2. *If $f(x)$ is a piecewise monotone function on* $[a, b]$, *then for any $\varepsilon>0$ there is some constant $b_{k,\varepsilon}>0$ such that for n sufficiently large $E_n^*(f) \leqq b_{k,\varepsilon}\omega(f; 1/n^{1-\varepsilon})$; i.e.,*

$$E_n^*(f) = o[\omega(f; 1/n^{1-\varepsilon})].$$

PROOF. Suppose $f(x)$ has k peaks. By (3), $E_n^*(S^p) \leqq (b-a)^{k+1}(C/n)^{p-k-1}$ whenever $n>2p$. If p is chosen so large that $(p-k-1)/(p+1)>1-\varepsilon$, then $\lambda_n=[E_n^*(S^p)]^{1/(p+1)}=o(1/n^{1-\varepsilon})$, and Theorem 2 then follows from Theorem 1.

THEOREM 3. *If $f(x)$ is a function such that for all x the kth difference $\Delta^k f(x) \geqq 0$ on* $[a, b]$, *then for any $\varepsilon>0$ there is some constant $d_{k,\varepsilon}>0$ such that for n sufficiently large*

$$E_{n,k}(f) \leqq d_{k,\varepsilon}\omega(f; 1/n^{1-\varepsilon}); \quad \text{i.e.,} \quad E_{n,k}(f) = o[\omega(f; 1/n^{1-\varepsilon})].$$

PROOF. Let

$$g(x) = 1/\lambda^{p+k+1}\int_x^{x+\lambda}\int_{t_{p+k}}^{t_{p+k}+\lambda}\int_{t_{p+k-1}}^{t_{p+k-1}+\lambda}\cdots\int_{t_1}^{t_1+\lambda} f(t)\,dt\,dt_1\cdots dt_{p+k},$$

where p and $\lambda>0$ will be specified later. Then

$$\|g-f\| \leqq \omega[f;(p+k+1)\lambda] \leqq (p+k+1)\omega(f;\lambda). \tag{8}$$

Applying the Fundamental Theorem of Calculus to $g(x)$ k times,

$$g^{(k)}(x) = \frac{1}{\lambda^{p+1}}\int_x^{x+\lambda}\int_{t_p}^{t_p+\lambda}\cdots\int_{t_1}^{t_1+\lambda}\frac{\Delta_\lambda^k f(t)}{\lambda^k}\,dt\,dt_1\cdots dt_p. \tag{9}$$

Since the integrand in (9) is assumed nonnegative, $g^{(k)}(x)\geqq 0$. Moreover, $|g^{(p+k+1)}(x)|\leqq 2^{p+k}\omega(f;\lambda)/\lambda^{p+k+1}$. Hence if $h(x)=\lambda^{p+k+1}g(x)/2^{p+k}\omega(f;\lambda)$, then $h(x)\in S^{p+k}$; then by Shisha's theorem (2) there is a polynomial $Q(x)\in\mathscr{P}_{n,k}$ such that

$$E_{n,k}(h) \leqq \|h-Q\| \leqq C_{p,k}/(n-p)^{p+1}.$$

Therefore, letting $P(x)=2^{p+k}\omega(f;\lambda)Q(x)/\lambda^{p+k+1}$, $P\in\mathscr{P}_{n,k}$ and

$$\|g-P\| \leqq C_{p,k}2^{p+k}\omega(f;\lambda)/\lambda^{p+k+1}(n-p)^{p+1}. \tag{10}$$

Hence, by (8) and (10),

$$\begin{aligned}\|f-P\| &\leqq \|f-g\| + \|g-P\|\\ &< \omega(f;\lambda)[p+k+1+C_{p,k}2^{p+k}/(n-p)^{p+1}\lambda^{p+k+1}].\end{aligned}$$

Choose p so large that $(p+1)/(p+k+1)>1-\varepsilon$, and then choose $\lambda=(n-p)^{-(p+1)/(p+k+1)}$. Then

$$\|f-P\| \leqq d_{k,\varepsilon}\omega(f;1/n^{1-\varepsilon}) \quad\text{and}\quad E_{n,k}(f) = o[\omega(f;1/n^{1-\varepsilon})]. \qquad \text{Q.E.D.}$$

REFERENCES

1. G. G. Lorentz and K. L. Zeller, *Degree of approximation by monotone polynomials*. I, J. Approximation Theory **1** (1968), 501–504. MR **39** #699.

2. D. J. Newman, E. Passow and L. Raymon, *Piecewise monotone polynomial approximation*, Trans. Amer. Math. Soc. **172** (1972), 465–472.

3. E. Passow and L. Raymon, *Comonotone polynomial approximation*, J. Approximation Theory (to appear).

4. J. A. Roulier, *Monotone approximation of certain classes of functions*, J. Approximation Theory **1** (1968), 319–324. MR **38** #4875.

5. O. Shisha, *Monotone approximation*, Pacific J. Math. **15** (1965), 667–671. MR **32** #2802.

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122