NON-\(n\)-MUTUALLY APOSYNDETIC CONTINUA

LELAND E. ROGERS

Abstract. Relationships are shown between non-\(n\)-mutual aposyndesis and \(C\)-cutting in compact metric continua, including results analogous to those of F. B. Jones in the case of nonaposyndesis.

1. Introduction. In [2], F. Burton Jones discussed nonaposyndesis in compact metric continua, including certain relationships between nonaposyndesis and both cut points and indecomposability. E. J. Vought [5] later proved the \(n\)-aposyndetic versions of many of Jones' results, as did C. L. Hagopian in the case of mutual aposyndesis [2]. This paper is concerned with the analogous results in the case of \(n\)-mutual aposyndesis [4], a generalization of both \(n\)-aposyndesis and mutual aposyndesis.

2. Definitions. A continuum is a nondegenerate closed connected set. The interior of a set \(A\) will be denoted by \(A^\circ\). If \(n \geq 2\) and \(A\) is an \(n\)-point subset of the continuum \(M\), then \(M\) is \(n\)-mutually aposyndetic at \(A\) if there exist \(n\) disjoint subcontinua of \(M\), each containing a point of \(A\) in its interior. If \(M\) is \(n\)-mutually aposyndetic at each \(n\)-point set, then \(M\) is said to be \(n\)-mutually aposyndetic. For \(x \in M\) and \(n \geq 2\), if there exists an \(n\)-point set \(A\) containing \(x\) such that \(M\) is not \(n\)-mutually aposyndetic at \(A\), then \(M\) is \(\text{non-}n\text{-mutually aposyndetic at } x\). For \(n \geq 2\), if \(M\) is \(n\)-mutually aposyndetic at each of its points, then \(M\) is totally \(n\)-mutually aposyndetic. If \(M\) is \(n\)-mutually aposyndetic at no \(n\)-point set, then \(M\) is strictly \(n\)-mutually aposyndetic. For \(n = 2\) we obtain the notions of mutual aposyndesis, total nonmutual aposyndesis, and strict nonmutual aposyndesis [2]. A set \(D\) is said to cut \(x\) from \(y\) in \(M\) if \(D\) intersects every subcontinuum of \(M\) which contains \(\{x, y\}\). A finite set \(\{p_1, \ldots, p_k\}\) is said to \(C\)-cut \(x\) from \(y\) if for each collection \(\{C_1, \ldots, C_k\}\) of disjoint subcontinua such that \(p_i \in C_i^\circ\) (for \(i \leq k\)), \(\bigcup_i C_i\) intersects each subcontinuum containing \(\{x, y\}\). For \(k = 1\) we obtain Hagopian's notion of a single point \(C\)-cutting [2, p. 618].

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3. **Preliminary theorems.** Theorems 1 and 2 correspond to Jones' Theorems 1 and 4 [3].

**Theorem 1.** Suppose that \( M \) is a regular Hausdorff continuum, \( n \geq 2 \), and that (1) for each \( i \geq 1 \), \( x_{1i}, \cdots, x_{ni} \) are distinct points such that \( M \) is not \( n \)-mutually aposyndetic at \( \{x_{ji}| j \leq n\} \), and (2) \( y_1, \cdots, y_n \) are distinct points of \( M \) such that for each \( j \leq n \), the sequence \( x_{j1}, x_{j2}, \cdots \) converges to \( y_j \). Then \( M \) is not \( n \)-mutually aposyndetic at \( \{y_j| j \leq n\} \).

**Proof.** Suppose that there are disjoint subcontinua \( H_1, \cdots, H_n \) such that for each \( j \leq n \), \( y_j \in H_j \). For each \( y_j \leq n \), let \( k_j \) be an integer such that if \( i \leq k_j \), then \( x_{ji} \in H_j \). Let \( k' = \max\{k_j| j \leq n\} \). Then for each \( j \leq n \), \( x_{jk'} \in H_j \). Hence \( M \) is \( n \)-mutually aposyndetic at \( \{x_{jk'}| j \leq n\} \), contrary to hypothesis. Thus the conclusion follows.

**Theorem 2.** Let \( n \geq 2 \). The set of points at which the compact metric continuum \( M \) is non-\( n \)-mutually aposyndetic is an \( F_\alpha \) set.

**Proof.** For each positive integer \( j \), let \( A_j \) be the set of all points \( x \in M \) such that there are distinct points \( p_1, \cdots, p_{n-1} \) in \( M - \{x\} \) satisfying the two properties that the distance between any pair in \( \{x\} \cup \{p_j| j \leq n-1\} \) is at least \( 1/j \), and that \( M \) is not \( n \)-mutually aposyndetic at \( \{x\} \cup \{p_j| j \leq n-1\} \).

It follows from Theorem 1 that each \( A_j \) is closed. Finally we observe that \( \bigcup_{j=1}^{n-1} A_j \) is exactly the set of points at which \( M \) is non-\( n \)-mutually aposyndetic. This completes the proof.

**Definition.** For \( n \geq 2 \) and an \( (n-1) \)-point set \( A \) in the continuum \( M \), \( D(A) \) denotes the set of all points \( x \) such that either \( x \in A \) or \( M \) is not \( n \)-mutually aposyndetic at \( A \cup \{x\} \).

It follows immediately from the definition that \( M \) is \( n \)-mutually aposyndetic if and only if for each \( (n-1) \)-point set \( A \), \( D(A) = A \). By Theorem 1, the set \( D(A) \) is always closed as is the case with the "aposyndetic" analog \( L_1 \) [3, p. 405]; but while \( L_1 \) is always connected, \( D(A) \) need not be connected. The following example shows that it may even be totally disconnected.

**Example (for \( n \geq 2 \)).** An \( (n-1) \)-mutually aposyndetic continuum which is not \( n \)-mutually aposyndetic on exactly one \( n \)-point set.

The continuum \( M \) will be constructed in \( E^3 \). For each \( i \geq 1 \) let \( T_i = [0, 1]^2 \times \{1/i\} \), and define \( T_0 = [0, 1]^2 \times \{0\} \). Let \( b_1, \cdots, b_{2n-2} \) be distinct points of \( \{1\} \times [0, 1] \times \{0\} \). For each \( j \leq 2n-2 \), let \( C_j = \{1\} \times \{y_j(b_j)\} \times [0, 1] \) (\( y_j \) is the projection map onto the \( y \)-axis). Thus each \( C_j \) meets each \( T_i \) and \( C_j \cap T_0 = \{b_j\} \). Let \( T = (\bigcup_{j=0}^{n-1} T_j \cup (\bigcup_{j=1}^{2n-2} C_j) \). Let \( y_1, \cdots, y_{n-1} \) be distinct points of the (two-dimensional) interior of \( T_1 \), and \( x \) a point of \( T_0 - \{b_j| j \leq 2n-2\} \). Let \( A_1, \cdots, A_{2n-2} \) be arcs lying in the (two-dimensional) interior of \( T_1 \), each pair intersecting in exactly the set \( \{y_j| j \leq n-1\} \).
and no arc crossing another. For each \( j \leq 2n - 2 \), let \( S_j \) be a homeomorph of \([0, 1]^2\) such that \( S_{jk} \cap T = \{x\} \cup \{b_i | i \neq j\} \cup A_j \). For \( j \leq 2n - 2 \) and \( k \geq 1 \), let \( S_{jk} \) be a homeomorph of \([0, 1]^2\) such that \( S_{jk} \cap T = \{x\} \cup \{b_i | i \neq j\} \) and such that for each \( j \), the sequence \( S_{jk}, S_{j2}, \ldots \) converges to \( S_{ja} \). Furthermore, we assume that the \( S_{jk}'s \) are chosen to be disjoint in the complement of \( T \). Finally we let

\[
M = \bigcup \{ S_{jk} \mid j \leq 2n - 2, k \geq 0 \} \cup T.
\]

Then \( M \) is \((n-1)\)-mutually aposyndetic, and \( M \) is not \( n \)-mutually aposyndetic at \( \{x\} \cup \{y_i | i \leq n-1\} \), but \( M \) is \( n \)-mutually aposyndetic at any other \( n \)-point set.

4. Cut point theorems. A compact metric continuum which is totally nonaposyndetic (i.e., aposyndetic at none of its points) must contain a cut point [3, p. 409]. In the case of total non-\( n \)-aposyndesis, there must exist an \( n \)-point set which cuts [1, p. 102]. However the corresponding result in the case of mutual aposyndesis does not hold even in the plane, since the example of [4, p. 241] can be observed to be a totally nonmutual aposyndetic continuum in which no point cuts. In fact even strict nonmutual aposyndesis does not guarantee existence of a cut point [2, p. 622]. However, the more general type of cutting, \( C \)-cutting, is guaranteed in the event of total nonmutual aposyndesis [2, p. 619]. This result is extended to the general case of \( n \geq 2 \) in a corollary to the following theorem.

**Theorem 3.** Suppose \( n \geq 2 \). Let \( U \) be an open set in the compact metric continuum \( M \), and \( L \) be a subset of \( M \) such that for each \( x \in U \) there exists an \((n-1)\)-point set \( A \subset L - \{x\} \) such that \( M \) is not \( n \)-mutually aposyndetic at \( \{x\} \cup A \). Then for each \( r \in M - L \), there exists a point \( s \in U \) such that, for each \((n-1)\)-point set \( B \cup L - \{s\} \) such that \( M \) is not \( n \)-mutually aposyndetic at \( \{s\} \cup B \), the set \( B \) must \( C \)-cut \( r \) from \( s \).

**Proof.** Let \( r \in M - L \). Suppose that the theorem fails and that \( \mathcal{G} \) denotes the collection of unions of \( n-1 \) disjoint continua missing \( r \), each containing a point of \( L \) in its interior.

Let \( s \in U \). Then there is an \((n-1)\)-point set \( A \subset L - \{s\} \) such that \( M \) is not \( n \)-mutually aposyndetic at \( \{s\} \cup A \) but \( A \) does not \( C \)-cut \( r \) from \( s \). Thus there are disjoint subcontinua \( C_1, \ldots, C_{n-1} \) each containing a point of \( A \) in its interior, and a continuum \( T \) such that \( \{r, s\} \subset T \) and \( T \cap (\bigcup_{i=1}^{n-1} C_i) = \emptyset \). Hence neither \( r \) nor \( s \) is in \( \bigcup_{i=1}^{n-1} C_i \). Since \( M \) is not \( n \)-mutually aposyndetic at \( s \cup A \), it follows that \( M \) must not be aposyndetic at \( s \) with respect to \( \bigcup_{i=1}^{n-1} C_i \). Note that \( \bigcup_{i=1}^{n-1} C_i \) is an element of the collection \( \mathcal{G} \).
Thus we have that for each $s \in U$, $M$ is not aposyndetic at $s$ with respect to some member of $\mathcal{G}$ which does not cut $r$ from $s$. But by [1, p. 101], there is a point $s \in U$ such that the associated $\bigcup C_i$ does cut $r$ from $s$. This contradiction concludes the proof of the theorem.

For $n=2$, Theorem 3 takes the form of Theorem 5 of [2, p. 618].

**Corollary 1.** Let $n \geq 2$. If no $(n-1)$-point set $C$-cuts in the compact metric continuum $M$, then $M$ is $n$-mutually aposyndetic at each point of a dense $G_\delta$ set.

**Proof.** Let $D$ be the set of points at which $M$ is $n$-mutually aposyndetic. By Theorem 2, $M-D$ is an $F_\sigma$ set; so $D$ is a $G_\delta$ set. Suppose that $D$ is not dense in $M$. Let $W$ be an open subset of $M-D$. For each positive integer $k$, let $A_k$ denote the set of all points $x \in W$ such that there exists an $(n-1)$-point set $B \subseteq M-\{x\}$ with the distance between any pair of points in $\{x\} \cup B$ not less than $1/k$, and with $M$ not $n$-mutually aposyndetic at $\{x\} \cup B$. By Theorem 1, each $A_k$ is closed relative to $W$. Note that $W = \bigcup_{k=1}^\infty A_k$. By the Baire category theorem, there is an integer $k'$ such that $A_{k'}$ has interior. Let $y \in A_{k'}$ and $\delta > 0$ such that $\delta < 1/k'$ and $N(y, \delta) \subseteq A_{k'}$ [the open ball of radius $d$ and center at $x$ is denoted by $N(x, d)$]. Let $r \in N(y, \delta/2)-N(y, \delta/4)$, and $L=M-\{r\}$. Then for each $x \in N(y, \delta/4)$, there is an $(n-1)$-point set $B \subseteq M-N(x, 1/k')$ such that $M$ is not $n$-mutually aposyndetic at $\{x\} \cup B$, and since the distance from $x$ to $r$ is at most $3\delta/4$ and $\delta \leq 1/k'$, we see that $B$ lies in $(M-\{r\})-\{x\}$ [which equals $L-\{x\}$]. Then by Theorem 3, there is a point $s \in N(y, \delta/4)$ such that if $B$ is an $(n-1)$-point set in $L-\{s\}$ and $M$ is not $n$-mutually aposyndetic at $\{s\} \cup B$, then $B$ must C-cut $r$ from $s$. Since $s \in A_{k'}$, there does exist an $(n-1)$-point set $B \subseteq M-N(s, 1/k')$ [which is contained in $(M-\{r\})-\{s\}=L-\{s\}$] such that $M$ is not $n$-mutually aposyndetic at $\{s\} \cup B$, and consequently $B$ must C-cut $r$ from $s$. This contradiction concludes the proof.

**Corollary 2.** Suppose $n \geq 2$. If the compact metric continuum $M$ is totally non-$n$-mutually aposyndetic, then $M$ contains an $(n-1)$-point set which $C$-cuts.

The next theorem is the $n$-mutual aposyndesis version of Theorem 17 of [3, p. 412].

**Theorem 4.** Let $n \geq 2$. Suppose the compact metric continuum $M$ is totally non-$n$-mutually aposyndetic and contains only one $(n-1)$-point set $N$ which $C$-cuts. Then for each $x \in M-N$, $M$ is not $n$-mutually aposyndetic at $\{x\} \cup N$. 

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Proof. Let \( x \in M - N \), and assume that \( M \) is \( n \)-mutually aposyndetic at \( \{x\} \cup N \). Let \( p_1, \ldots, p_{n-1} \) denote the elements of \( N \). Then there are disjoint continua \( K, H_1, \ldots, H_{n-1} \) such that \( x \in K^0 \) and \( p_i \in H_i^0 \) for each \( i \leq n-1 \). For each \( y \in K^0 \), \( M \) is \( n \)-mutually aposyndetic at \( \{y\} \cup N \). Hence for each such \( y \), there is an \((n-1)\)-point set \( J_y \) different from \( N \) such that \( M \) is not \( n \)-mutually aposyndetic at \( \{y\} \cup J_y \). For each \( i \leq n-1 \) and each \( j \geq 1 \), let \( A_{ij} \) be the set of all points \( y \in K^0 \) such that \( p_i \notin J_y \) and the distance between each pair of points in \( \{p_i\} \cup J_y \) is at least \( \|j \). By Theorem 1, each \( A_{ij} \) is closed relative to \( K^0 \). Since \( K^0 = \bigcup \{A_{ij}|i \leq n-1, j \geq 1\} \), by the Baire category theorem, some \( A_{ij} \) has interior. Then by Theorem 3, there is a point \( s \in A_{ij} \) and corresponding \( J_s \) that \( C \)-cuts \( p_i \) from \( s \). But since \( J_s \neq N \) and \( N \) was the only \((n-1)\)-point set which \( C \)-cuts, we have a contradiction.

Using the following modified concept of composants due to Hagopian \[2, p. 620\] we obtain an analog to Theorem 16 of \[3, p. 411\].

Definition. The \( p \)-quasi-composant of the continuum \( M \) is the set consisting of \( p \) together with the union of all subcontinua containing \( p \) and missing some subcontinuum with interior.

Theorem 5. If the continuum \( M \) has only one \( C \)-cut point \( p \), then the \( p \)-quasi-composant of \( M \) is all of \( M \).

Proof. Let \( x \) and \( y \) be points of \( M - \{p\} \). Since \( p \) is the only \( C \)-cut point, \( x \) cannot \( C \)-cut \( y \) from \( p \), so there are disjoint continua \( H \) and \( K \) such that \( x \in H^0 \) and \( \{p, y\} \subset K \). Thus \( y \in p \)-quasi-composant of \( M \). Since \( y \) was arbitrary in \( M - \{p\} \), we have that \( M = p \)-quasi-composant.

Example. A totally nonmutually aposyndetic compact metric continuum which contains exactly one \( C \)-cut point.

The set of all nonzero integers will be denoted by \( \mathbb{Z}' \). For each \( n \in \mathbb{Z}' \), let \( a_n = 1/(2n\pi + \pi/6) \) and \( b_n = 1/2n\pi \). Let

\[
K = \{ (0, y) \mid -1 \leq y \leq 1 \} \cup \{ (x, \sin 1/x) \mid 0 < |x| \leq 1/\pi \}
\]

with the two points \((-1/\pi, 0)\) and \((1/\pi, 0)\) identified. Set

\[
K' = K \cup ( \bigcup \{ (b_n, y) \mid 0 \leq y \leq 1/2, n \in \mathbb{Z}' \} )
\]

\[
\cup ( \bigcup \{ (x, \frac{1}{2}) \mid a_n \leq x \leq b_n, n \in \mathbb{Z}' \} ).
\]

Let \( A \) and \( B \) be the following subsets of \( K' \times [0, 1] \):

\[
A = \bigcup \{ (x, \frac{1}{2}, z) \mid a_n < x < b_n, 0 \leq z < (b_n - x)/(b_n - a_n), n \in \mathbb{Z}' \},
\]

\[
B = \bigcup \{ (x, \sin 1/x, z) \mid a_n < x < b_n, 2 |z - \frac{1}{2}| < (x - a_n)/(b_n - a_n), n \in \mathbb{Z}' \}.
\]
Let \( K'' = K' \times [0, 1] - (A \cup B) \). With the Cantor set denoted by \( C \), we define \( K''' = K'' \times C \) with the set \( \{(0, y, z)\} \times C \) identified for each pair \( (y, z) \in [0, 1]^p \), i.e., the Cantor set of limiting (unit-square) disks are identified to form one limiting disk. Finally, let \( M \) denote the continuum \( K''' \) with the four corners of the limiting disk identified to form a point \( p \). Then \( M \) is totally nonmutually aposyndetic and has only one \( C \)-cut point, namely \( p \).

**Theorem 6.** If the set of all \( C \)-cut points in a compact planar continuum \( M \) is totally disconnected, then \( M \) is locally connected.

**Proof.** Suppose that \( M \) is not locally connected. Then by [5, p. 130], \( M \) is not 2-aposyndetic. Thus there are distinct points \( x, y, z \in M \) such that \( M \) is not aposyndetic at \( x \) with respect to \( \{y, z\} \). Let \( L \) denote the set of all points \( p \) such that \( M \) is not aposyndetic at \( p \) with respect to \( \{y, z\} \). Note that \( \{x, y, z\} \subseteq L \). Since \( L \) has at most two components [5, p. 128], there must be a nondegenerate continuum \( K \) contained in \( L \). For each \( p \in K - \{y, z\} \), \( p \) \( C \)-cuts \( y \) from \( z \). It follows that the set of all \( C \)-cut points is not totally disconnected. This concludes the proof.

**Theorem 7.** Let \( n \geq 2 \). The regular Hausdorff continuum \( M \) is strictly non-\( n \)-mutually aposyndetic if and only if for each set \( \{p_1, \cdots, p_n-1\} \) of \( n-1 \) points and each open set \( U \), there exist points \( r, s \in U \) such that \( \{p_i | i < n\} \) \( C \)-cuts \( r \) from \( s \).

**Proof.** Assume that \( M \) is strictly non-\( n \)-mutually aposyndetic. Let \( p_1, \cdots, p_{n-1} \) be distinct points of \( M \), and let \( U \) be an open set. For each \( x \in U \), \( M \) is not \( n \)-mutually aposyndetic at \( \{x\} \cup \{p_i | i < n\} \). Let \( r \in U - \{p_i | i < n\} \). Then by Theorem 3, there is a point \( s \in U \) such that \( \{p_i | i < n\} \) \( C \)-cuts \( r \) from \( s \).

To prove the converse, we suppose to the contrary that \( x_1, \cdots, x_n \) are distinct points and \( M \) is \( n \)-mutually aposyndetic at \( \{x_i | i \leq n\} \). Then there are disjoint subcontinua \( C_1, \cdots, C_n \) with \( x_i \in C_i^o \) (for each \( i \leq n \)). Consequently, for each pair of points \( r, s \) in the open set \( C_n^o \), \( \{x_i | i \leq n-1\} \) does not \( C \)-cut \( r \) from \( s \). Thus the proof is complete.

Thus we see that while a totally non-\( n \)-mutually continuum may contain only one \( C \)-cut set (of \( n-1 \) points), in strictly non-\( n \)-mutually aposyndetic continua every \( (n-1) \)-point set \( C \)-cuts.

**References**


Department of Mathematics, University of Wyoming, Laramie, Wyoming 82070