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NON-n-MUTUALLY APOSYNDETIC CONTINUA

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ABSTRACT. Relationships are shown between non-n-mutual aposyndesis and C-cutting in compact metric continua, including results analogous to those of F. B. Jones in the case of nonaposyndesis.

1. **Introduction.** In [2], F. Burton Jones discussed nonaposyndesis in compact metric continua, including certain relationships between nonaposyndesis and both cut points and indecomposability.

E. J. Vought [5] later proved the n-aposyndetic versions of many of Jones' results, as did C. L. Hagopian in the case of mutual aposyndesis [2]. This paper is concerned with the analogous results in the case of n-mutual aposyndesis [4], a generalization of both n-aposyndesis and mutual aposyndesis.

2. **Definitions.** A *continuum* is a nondegenerate closed connected set. The interior of a set A will be denoted by A^o. If $n \geqq 2$ and A is an n-point subset of the continuum M, then M is *n-mutually aposyndetic at A* if there exist n disjoint subcontinua of M, each containing a point of A in its interior. If M is n-mutually aposyndetic at each n-point set, then M is said to be *n-mutually aposyndetic*. For $x \in M$ and $n \geqq 2$, if there exists an n-point set A containing x such that M is not n-mutually aposyndetic at A, then M is *non-n-mutually aposyndetic at x*. For $n \geqq 2$, if M is non-n-mutually aposyndetic at each of its points, then M is *totally non-n-mutually aposyndetic*. If M is n-mutually aposyndetic at no n-point set, then M is *strictly non-n-mutually aposyndetic*. For $n=2$ we obtain the notions of mutual aposyndesis, total nonmutual aposyndesis, and strict nonmutual aposyndesis [2]. A set D is said to *cut x* from y in M if D intersects every subcontinuum of M which contains $\{x, y\}$. A finite set $\{p_1, \cdots, p_k\}$ is said to *C-cut x* from y if for each collection $\{C_1, \cdots, C_k\}$ of disjoint subcontinua such that $p_i \in C_i^o$ (for $i \leqq k$), $\bigcup_1^k C_i$ intersects each subcontinuum containing $\{x, y\}$. For $k=1$ we obtain Hagopian's notion of a single point C-cutting [2, p. 618].

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3. **Preliminary theorems.** Theorems 1 and 2 correspond to Jones' Theorems 1 and 4 [3].

THEOREM 1. *Suppose that M is a regular Hausdorff continuum, $n \geqq 2$, and that* (1) *for each $i \geqq 1$, $x_{1i}, \cdots, x_{ni}$ are distinct points such that M is not n-mutually aposyndetic at $\{x_{ji} | j \leqq n\}$, and* (2) *$y_1, \cdots, y_n$ are distinct points of M such that for each $j \leqq n$, the sequence $x_{j1}, x_{j2}, \cdots$ converges to y_j. Then M is not n-mutually aposyndetic at $\{y_j | j \leqq n\}$.*

PROOF. Suppose that there are disjoint subcontinua $H_1, \cdots, H_n$ such that for each $j \leqq n$, $y_j \in H_j^o$. For each $j \leqq n$, let k_j be an integer such that if $i \geqq k_j$ then $x_{ji} \in H_j^o$. Let $k' = \max\{k_j | j \leqq n\}$. Then for each $j \leqq n$, $x_{jk'} \in H_j^o$. Hence M is n-mutually aposyndetic at $\{x_{jk'} | j \leqq n\}$, contrary to hypothesis. Thus the conclusion follows.

THEOREM 2. *Let $n \geqq 2$. The set of points at which the compact metric continuum M is non-n-mutually aposyndetic is an F_σ set.*

PROOF. For each positive integer j, let A_j be the set of all points $x \in M$ such that there are distinct points $p_1, \cdots, p_{n-1}$ in $M - \{x\}$ satisfying the two properties that the distance between any pair in $\{x\} \cup \{p_i | i \leqq n-1\}$ is at least $1/j$, and that M is not n-mutually aposyndetic at $\{x\} \cup \{p_i | i \leqq n-1\}$. It follows from Theorem 1 that each A_j is closed. Finally we observe that $\bigcup_1^\infty A_j$ is exactly the set of points at which M is non-n-mutually aposyndetic. This completes the proof

DEFINITION. For $n \geqq 2$ and an $(n-1)$-point set A in the continuum M, $D(A)$ denotes the set of all points x such that either $x \in A$ or M is not n-mutually aposyndetic at $A \cup \{x\}$.

It follows immediately from the definition that M is n-mutually aposyndetic if and only if for each $(n-1)$-point set A, $D(A) = A$. By Theorem 1, the set $D(A)$ is always closed as is the case with the "aposyndetic" analog L_x [3, p. 405]; but while L_x is always connected, $D(A)$ need not be connected. The following example shows that it may even be totally disconnected.

EXAMPLE (FOR $n \geqq 2$). *An $(n-1)$-mutually aposyndetic continuum which is not n-mutually aposyndetic on exactly one n-point set.*

The continuum M will be constructed in E^3. For each $i \geqq 1$ let $T_i = [0, 1]^2 \times \{1/i\}$, and define $T_0 = [0, 1]^2 \times \{0\}$. Let $b_1, \cdots, b_{2n-2}$ be distinct points of $\{1\} \times [0, 1] \times \{0\}$. For each $j \leqq 2n-2$, let $C_j = \{1\} \times \{\pi_2(b_j)\} \times [0, 1]$ (π_2 is the projection map onto the y-axis). Thus each C_j meets each T_i and $C_j \cap T_0 = \{b_j\}$. Let $T = (\bigcup_0^\infty T_i) \cup (\bigcup_1^{2n-2} C_i)$. Let $y_1, \cdots, y_{n-1}$ be distinct points of the (two-dimensional) interior of T_1, and x a point of $T_0 - \{b_i | i \leqq 2n-2\}$. Let $A_1, \cdots, A_{2n-2}$ be arcs lying in the (two-dimensional) interior of T_1, each pair intersecting in exactly the set $\{y_i | i \leqq n-1\}$

and no arc crossing another. For each $j \leqq 2n-2$, let S_j be a homeomorph of $[0, 1]^2$ such that $S_{j0} \cap T = \{x\} \cup \{b_i | i \neq j\} \cup A_j$. For $j \leqq 2n-2$ and $k \geqq 1$, let S_{jk} be a homeomorph of $[0, 1]^2$ such that $S_{jk} \cap T = \{x\} \cup \{b_i | i \neq j\}$ and such that for each j, the sequence $S_{j1}, S_{j2}, \cdots$ converges to S_{j0}. Furthermore, we assume that the S_{jk}'s are chosen to be disjoint in the complement of T. Finally we let

$$M = \bigcup \{S_{jk} \,|\, j \leqq 2n-2, k \geqq 0\} \cup T.$$

Then M is $(n-1)$-mutually aposyndetic, and M is not n-mutually aposyndetic at $\{x\} \cup \{y_i | i \leqq n-1\}$, but M *is* n-mutually aposyndetic at any other n-point set.

4. **Cut point theorems.** A compact metric continuum which is totally nonaposyndetic (i.e., aposyndetic at none of its points) must contain a cut point [**3**, p. 409]. In the case of total non-n-aposyndesis, there must exist an n-point set which cuts [**1**, p. 102]. However the corresponding result in the case of mutual aposyndesis does not hold even in the plane, since the example of [**4**, p. 241] can be observed to be a totally nonmutually aposyndetic continuum in which no point cuts. In fact even strict nonmutual aposyndesis does not guarantee existence of a cut point [**2**, p. 622]. However, the more general type of cutting, C-cutting, is guaranteed in the event of total nonmutual aposyndesis [**2**, p. 619]. This result is extended to the general case of $n \geqq 2$ in a corollary to the following theorem.

THEOREM 3. *Suppose $n \geqq 2$. Let U be an open set in the compact metric continuum M, and L be a subset of M such that for each $x \in U$ there exists an $(n-1)$-point set $A \subset L - \{x\}$ such that M is not n-mutually aposyndetic at $\{x\} \cup A$. Then for each $r \in M - L$, there exists a point $s \in U$ such that, for each $(n-1)$-point set $B \cup L - \{s\}$ such that M is not n-mutually aposyndetic at $\{s\} \cup B$, the set B must C-cut r from s.*

PROOF. Let $r \in M - L$. Suppose that the theorem fails and that $\mathcal{G}$ denotes the collection of unions of $n-1$ disjoint continua missing r, each containing a point of L in its interior.

Let $s \in U$. Then there is an $(n-1)$-point set $A \subset L - \{s\}$ such that M is not n-mutually aposyndetic at $\{s\} \cup A$ *but* A does not C-cut r from s. Thus there are disjoint subcontinua $C_1, \cdots, C_{n-1}$ each containing a point of A in its interior, and a continuum T such that $\{r, s\} \subset T$ and $T \cap (\bigcup_1^{n-1} C_i) = \varnothing$. Hence neither r nor s is in $\bigcup_1^{n-1} C_i$. Since M is not n-mutually aposyndetic at $s \cup A$, it follows that M must not be aposyndetic at s with respect to $\bigcup_1^{n-1} C_i$. Note that $\bigcup_1^{n-1} C_i$ is an element of the collection $\mathcal{G}$.

Thus we have that for each $s \in U$, M is not aposyndetic at s with respect to some member of $\mathcal{G}$ which does not cut r from s. But by [**1**, p. 101], there is a point $s \in U$ such that the associated $\bigcup C_i$ *does* cut r from s. This contradiction concludes the proof of the theorem.

For $n=2$, Theorem 3 takes the form of Theorem 5 of [**2**, p. 618].

COROLLARY 1. *Let $n \geqq 2$. If no $(n-1)$-point set C-cuts in the compact metric continuum M, then M is n-mutually aposyndetic at each point of a dense G_δ set.*

PROOF. Let D be the set of points at which M is n-mutually aposyndetic. By Theorem 2, $M-D$ is an F_σ set; so D is a G_δ set. Suppose that D is not dense in M. Let W be an open subset of $M-D$. For each positive integer k, let A_k denote the set of all points $x \in W$ such that there exists an $(n-1)$-point set $B \subset M-\{x\}$ with the distance between any pair of points in $\{x\} \cup B$ not less than $1/k$, and with M not n-mutually aposyndetic at $\{x\} \cup B$. By Theorem 1, each A_k is closed relative to W. Note that $W = \bigcup_{k=1}^{\infty} A_k$. By the Baire category theorem, there is an integer k' such that $A_{k'}$ has interior. Let $y \in A_{k'}^{\circ}$ and $\delta > 0$ such that $\delta < 1/k'$ and $N(y, \delta) \subset A_{k'}^{\circ}$ [the open ball of radius d and center at x is denoted by $N(x, d)$]. Let $r \in N(y, \delta/2) - N(y, \delta/4)$, and $L = M - \{r\}$. Then for each $x \in N(y, \delta/4)$, there is an $(n-1)$-point set $B \subset M - N(x, 1/k')$ such that M is not n-mutually aposyndetic at $\{x\} \cup B$, and since the distance from x to r is at most $3\delta/4$ and $\delta \leqq 1/k'$, we see that B lies in $(M-\{r\})-\{x\}$ [which equals $L-\{x\}$]. Then by Theorem 3, there is a point $s \in N(y, \delta/4)$ such that if B is an $(n-1)$-point set in $L-\{s\}$ and M is not n-mutually aposyndetic at $\{s\} \cup B$, then B must C-cut r from s. Since $s \in A_{k'}$, there does exist an $(n-1)$-point set $B \subset M - N(s, 1/k')$ [which is contained in $(M-\{r\})-\{s\} = L-\{s\}$] such that M is not n-mutually aposyndetic at $\{s\} \cup B$, and consequently B must C-cut r from s. This contradiction concludes the proof.

COROLLARY 2. *Suppose $n \geqq 2$. If the compact metric continuum M is totally non-n-mutually aposyndetic, then M contains an $(n-1)$-point set which C-cuts.*

The next theorem is the n-mutual aposyndesis version of Theorem 17 of [**3**, p. 412].

THEOREM 4. *Let $n \geqq 2$. Suppose the compact metric continuum M is totally non-n-mutually aposyndetic and contains only one $(n-1)$-point set N which C-cuts. Then for each $x \in M-N$, M is not n-mutually aposyndetic at $\{x\} \cup N$.*

PROOF. Let $x \in M-N$, and assume that M is n-mutually aposyndetic at $\{x\} \cup N$. Let $p_1, \cdots, p_{n-1}$ denote the elements of N. Then there are disjoint continua $K, H_1, \cdots, H_{n-1}$ such that $x \in K^o$ and $p_i \in H_i^o$ for each $i \leqq n-1$. For each $y \in K^o$, M is n-mutually aposyndetic at $\{y\} \cup N$. Hence for each such y, there is an $(n-1)$-point set J_y different from N such that M is not n-mutually aposyndetic at $\{y\} \cup J_y$. For each $i \leqq n-1$ and each $j \geqq 1$, let A_{ij} be the set of all points $y \in K^o$ such that $p_i \notin J_y$ and the distance between each pair of points in $\{p_i\} \cup J_y$ is at least $1/j$. By Theorem 1, each A_{ij} is closed relative to K^o. Since $K^o = \bigcup \{A_{ij} | i \leqq n-1, j \geqq 1\}$, by the Baire category theorem, some $A_{i'j'}$ has interior. Then by Theorem 3, there is a point $s \in A_{i'j'}^o$ and corresponding J_s that C-cuts $p_{i'}$ from s. But since $J_s \neq N$ and N was the only $(n-1)$-point set which C-cuts, we have a contradiction.

Using the following modified concept of composants due to Hagopian [**2**, p. 620] we obtain an analog to Theorem 16 of [**3**, p. 411].

DEFINITION. The *p-quasi-composant* of the continuum M is the set consisting of p together with the union of all subcontinua containing p and missing some subcontinuum with interior.

THEOREM 5. *If the continuum M has only one C-cut point p, then the p-quasi-composant of M is all of M.*

PROOF. Let x and y be points of $M-\{p\}$. Since p is the only C-cut point, x cannot C-cut y from p, so there are disjoint continua H and K such that $x \in H^o$ and $\{p, y\} \subset K$. Thus $y \in p$-quasi-composant of M. Since y was arbitrary in $M-\{p\}$, we have that $M = p$-quasi-composant.

EXAMPLE. *A totally nonmutually aposyndetic compact metric continuum which contains exactly one C-cut point.*

The set of all nonzero integers will be denoted by Z'. For each $n \in Z'$, let $a_n = 1/(2n\pi + \pi/6)$ and $b_n = 1/2n\pi$. Let

$$K = \{(0, y) \mid -1 \leqq y \leqq 1\} \cup \{(x, \sin 1/x) \mid 0 < |x| \leqq 1/\pi\}$$

with the two points $(-1/\pi, 0)$ and $(1/\pi, 0)$ identified. Set

$$\begin{aligned} K' = K &\cup \left(\bigcup \{(b_n, y) \mid 0 \leqq y \leqq \tfrac{1}{2}, n \in Z'\}\right) \\ &\cup \left(\bigcup \{(x, \tfrac{1}{2}) \mid a_n \leqq x \leqq b_n, n \in Z'\}\right). \end{aligned}$$

Let A and B be the following subsets of $K' \times [0, 1]$:

$$A = \bigcup \{(x, \tfrac{1}{2}, z) \mid a_n < x < b_n, 0 \leqq z < (b_n - x)/(b_n - a_n), n \in Z'\},$$

$$\begin{aligned} B = \bigcup \{(x, \sin 1/x, z) \mid a_n &< x < b_n, \\ &2|z - \tfrac{1}{2}| < (x - a_n)/(b_n - a_n), n \in Z'\}. \end{aligned}$$

Let $K''=K'\times[0, 1]-(A\cup B)$. With the Cantor set denoted by C, we define $K'''=K''\times C$ with the set $\{(0, y, z)\}\times C$ identified for each pair $(y, z)\in[0, 1]^2$, i.e., the Cantor set of limiting (unit-square) disks are identified to form one limiting disk. Finally, let M denote the continuum K''' with the four corners of the limiting disk identified to form a point p. Then M is totally nonmutually aposyndetic and has only one C-cut point, namely p.

THEOREM 6. *If the set of all C-cut points in a compact planar continuum M is totally disconnected, then M is locally connected.*

PROOF. Suppose that M is not locally connected. Then by [**5**, p. 130], M is not 2-aposyndetic. Thus there are distinct points x, y, $z\in M$ such that M is not aposyndetic at x with respect to $\{y, z\}$. Let L denote the set of all points p such that M is not aposyndetic at p with respect to $\{y, z\}$. Note that $\{x, y, z\}\subset L$. Since L has at most two components [**5**, p. 128], there must be a nondegenerate continuum K contained in L. For each $p\in K-\{y, z\}$, p C-cuts y from z. It follows that the set of all C-cut points is not totally disconnected. This concludes the proof.

THEOREM 7. *Let $n\geqq 2$. The regular Hausdorff continuum M is strictly non-n-mutually aposyndetic if and only if for each set $\{p_1, \cdots, p_{n-1}\}$ of $n-1$ points and each open set U, there exist points r, $s\in U$ such that $\{p_i|i<n\}$ C-cuts r from s.*

PROOF. Assume that M is strictly non-n-mutually aposyndetic. Let $p_1, \cdots, p_{n-1}$ be distinct points of M, and let U be an open set. For each $x\in U$, M is not n-mutually aposyndetic at $\{x\}\cup\{p_i|i<n\}$. Let $r\in U-\{p_i|i<n\}$. Then by Theorem 3, there is a point $s\in U$ such that $\{p_i|i<n\}$ C-cuts r from s.

To prove the converse, we suppose to the contrary that $x_1, \cdots, x_n$ are distinct points and M *is* n-mutually aposyndetic at $\{x_i|i\leqq n\}$. Then there are disjoint subcontinua $C_1, \cdots, C_n$ with $x_i\in C_i^o$ (for each $i\leqq n$). Consequently, for each pair of points r, s in the open set C_n^o, $\{x_i|i\leqq n-1\}$ does not C-cut r from s. Thus the proof is complete.

Thus we see that while a totally non-n-mutually continuum may contain only one C-cut set (of $n-1$ points), in strictly non-n-mutually aposyndetic continua *every* $(n-1)$-point set C-cuts.

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