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EVERY FINITELY GENERATED FISCHER GROUP IS FINITE

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ABSTRACT. By a simple combinatorial argument, and using a result of Bruck, the Burnside problem is settled affirmatively for the class of Fischer 3-groups.

0. **Introduction.** In a recent book on cubic forms by Yu. I. Manin [1] the Burnside problem for the class of Fischer 3-groups was posed (cf. p. 39 of [1]). In this note we give a simple combinatorial argument leading to an affirmative settlement of this problem. A result of H. R. Bruck [2] that every finitely generated commutative Moufang loop of period 3 is finite, is essential to our proof. We also need some theorems connecting symmetric quasigroups, Moufang loops and Fischer groups.

1. **Definitions.** A *Fischer group* is a pair (G, E) such that (i) G is a group, (ii) E is a generating subset of G, and (iii) $x^2=1$, $(xy)^3=1$ and $xyx \in E$ for all $x, y \in E$.

It is easy to see that E is a complete conjugacy class of G and that the commutator subgroup G' is generated by the elements of the form xy, $(x, y \in E)$. If we define the following operation on E

$$x \circ y = xyx \tag{1}$$

then $(E, \circ)$ becomes a *distributive symmetric quasigroup*, i.e. we have $x \circ y=y \circ x$, $x \circ (x \circ y)=y$, $x \circ (y \circ z)=(x \circ y) \circ (x \circ z)$ $(x, y, z \in E)$. Moreover if we fix $u \in E$ and define

$$x * y = u \circ (x \circ y), \qquad x, y \in E,$$

we make $(E, *)$ a *commutative Moufang loop*, i.e. all the axioms for an abelian group hold except associativity, which is replaced by *weak associativity*

$$(x * y) * (x * z) = x^2 * (y * z) \qquad (x, y, z \in E, x^2 = x * x).$$

2. **Some auxiliary results.** We mention here some results needed in the proof of our main theorem.

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2.1 THEOREM (BRUCK [2]). *Every finitely generated commutative Moufang loop of period* 3 (*i.e.* $x^3=1$ *if* x *is an element of the loop*) *is finite.*

2.2 PROPOSITION (BENKOV-BELOUSOV [4]). *If* E *is a distributive symmetric quasigroup and* $u \in E$ *is fixed, then* $(E, *)$ *is a commutative Moufang loop of period* 3 *with identity element* u, *where* $x * y = u \circ (x \circ y)$. *Conversely, if* E *is a commutative Moufang loop of period* 3 *and* c *is in the associative center of* E (*i.e.* $c * (x * y) = (c * x) * y$ *for all* $x, y \in E$), *then* $(E, \circ)$ *is a distributive symmetric quasigroup where* $x \circ y = c * x^{-1} * y^{-1}$.

2.3 PROPOSITION (CF. [1, THEOREM 3.8]). *Let* (G, E) *be a Fischer group and let* $(E, \circ)$ *be the distributive symmetric quasigroup with operation defined by* (1). *If* $T(E)$ *denotes the group generated by the permutations* $(t_x)_{x\in E}$, $t_x(y) = x \circ y$, *then the map* $f: E \to T(E)$ *defined by* $x \to t_x$ *can be extended to* G *as a group homomorphism and* $\ker f = Z(G) =$ *the center of* G. *Thus* $(T(E), E)$ *is a Fischer group and* $T(E) \simeq G/Z(G)$. *Here we identify* $x \in E$ *with* $t_x \in T(E)$.

3. **Statement of the problem.** We say that a set I generates a Fischer group (G, E) if $I \subset E$ and I generates G. If I is finite, is G finite? Let us construct the free Fischer group generated by I. Let $F(I)$ be the free group generated by I, N the normal subgroup of $F(I)$ generated by the words x^2, $(s_1 x s_1^{-1} s_2 y s_2^{-1})^3$, $x, y \in I$, $s_1, s_2 \in F(I)$, $G(I) = F(I)/N$, $f: F(I) \to G(I)$ (the canonical map) and $E(I) \subset G(I)$ the conjugacy class containing $f(I)$. Then $(G(I), E(I))$ is the *free Fischer group* generated by I and $E(I)$ the *free distributive symmetric quasigroup* generated by I. By using 2.1 and 2.2 we see that $E(I)$ is finite if I is finite. But unfortunately 2.3 tells us only that $G(I)/Z(G(I))$ is finite.

4. We now proceed to establishing our result. We start with the following combinatorial.

4.1 PROPOSITION. *Let* (G, E) *be a Fischer group with* $\#E = n$. *Then*

$$\#G \leqq \sum_{k=0}^{m} \prod_{i=1}^{k} (n - \tfrac{1}{2}i(i-1))$$
$$= 1 + n + n(n-1) + \cdots + \prod_{i=1}^{m} (n - \tfrac{1}{2}i(i-1))$$

where $m = \max\{k \in N : \frac{1}{2}k(k-1) \leqq n\}$.

PROOF. Let W_k denote the irreducible words of length k with letters in E. Then $\#W_0 = 1$, $\#W_1 = n$. We claim that

$$\#W_k \leqq \prod_{i=1}^{k} (n - \tfrac{1}{2}i(i-1)) \quad \text{if } \tfrac{1}{2}k(k-1) \leqq n, \qquad k = 1, 2, \cdots.$$

Indeed, let $x_1x_2x_3 \cdots x_k \in W_k$. Then for x_1 there are n possible choices, for x_2 there are $n-1$ possible choices ($x_2 \neq x_1$, because otherwise the length of the word could be reduced by 2 since $x^2=1$), for x_3 we have $n-3$ possible choices ($x_3 \neq x_1$, x_2, and $x_1x_2x_1$), for x_4, $n-6$ choices ($x_4 \neq x_1$, x_2, x_3, $x_1x_2x_1$, $x_1x_3x_1$, $x_2x_3x_2$) etc. Thus the total number of words with irreducible length k is at most $\prod_{i=1}^{k} (n-\frac{1}{2}i(i-1))$. If $k \geqq 2$ this is a pretty liberal estimate, e.g. if $xyx=z \in E$, then $xy=yz=zx$ etc.

4.2 Theorem. *Every finitely generated Fischer group is finite.*

Proof. Suppose that the group is generated by a finite set I. It is enough to prove that the free Fischer group $(G(I), E(I))$ is finite. By Proposition 2.2 and Theorem 2.1, $E(I)$ is finite. Therefore $G(I)$ is finite by Proposition 4.1.

5. **Remarks.** Fischer groups were defined in [3] in a more general way involving a general prime p instead of 3. By a theorem due to Fischer (under finiteness assumption) it is obtained that $\#E=3^k$, $\#G=2 \cdot 3^m$ and G is solvable. It would be interesting to know the exact order of G if we know that $\#E=3^k$ and no other relations are introduced except those warranted by the definition of a Fischer group. For example it is easy to see that $\#W_2=3^{k-1}(3^k-1)$.

References

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