

EVERY FINITELY GENERATED FISCHER GROUP IS FINITE

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ABSTRACT. By a simple combinatorial argument, and using a result of Bruck, the Burnside problem is settled affirmatively for the class of Fischer 3-groups.

0. Introduction. In a recent book on cubic forms by Yu. I. Manin [1] the Burnside problem for the class of Fischer 3-groups was posed (cf. p. 39 of [1]). In this note we give a simple combinatorial argument leading to an affirmative settlement of this problem. A result of H. R. Bruck [2] that every finitely generated commutative Moufang loop of period 3 is finite, is essential to our proof. We also need some theorems connecting symmetric quasigroups, Moufang loops and Fischer groups.

1. Definitions. A *Fischer group* is a pair (G, E) such that (i) G is a group, (ii) E is a generating subset of G , and (iii) $x^2=1$, $(xy)^3=1$ and $xyx \in E$ for all $x, y \in E$.

It is easy to see that E is a complete conjugacy class of G and that the commutator subgroup G' is generated by the elements of the form xy , $(x, y \in E)$. If we define the following operation on E

$$(1) \quad x \circ y = xyx$$

then (E, \circ) becomes a *distributive symmetric quasigroup*, i.e. we have $x \circ y = y \circ x$, $x \circ (x \circ y) = y$, $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$ ($x, y, z \in E$). Moreover if we fix $u \in E$ and define

$$x * y = u \circ (x \circ y), \quad x, y \in E,$$

we make $(E, *)$ a *commutative Moufang loop*, i.e. all the axioms for an abelian group hold except associativity, which is replaced by *weak associativity*

$$(x * y) * (x * z) = x^2 * (y * z) \quad (x, y, z \in E, x^2 = x * x).$$

2. Some auxiliary results. We mention here some results needed in the proof of our main theorem.

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2.1 THEOREM (BRUCK [2]). *Every finitely generated commutative Moufang loop of period 3 (i.e. $x^3=1$ if x is an element of the loop) is finite.*

2.2 PROPOSITION (BENKOV-BELOUSOV [4]). *If E is a distributive symmetric quasigroup and $u \in E$ is fixed, then $(E, *)$ is a commutative Moufang loop of period 3 with identity element u , where $x * y = u \circ (x \circ y)$. Conversely, if E is a commutative Moufang loop of period 3 and c is in the associative center of E (i.e. $c * (x * y) = (c * x) * y$ for all $x, y \in E$), then (E, \circ) is a distributive symmetric quasigroup where $x \circ y = c * x^{-1} * y^{-1}$.*

2.3 PROPOSITION (CF. [1, THEOREM 3.8]). *Let (G, E) be a Fischer group and let (E, \circ) be the distributive symmetric quasigroup with operation defined by (1). If $T(E)$ denotes the group generated by the permutations $(t_x)_{x \in E}$, $t_x(y) = x \circ y$, then the map $f: E \rightarrow T(E)$ defined by $x \rightarrow t_x$ can be extended to G as a group homomorphism and $\ker f = Z(G) =$ the center of G . Thus $(T(E), E)$ is a Fischer group and $T(E) \cong G/Z(G)$. Here we identify $x \in E$ with $t_x \in T(E)$.*

3. **Statement of the problem.** We say that a set I generates a Fischer group (G, E) if $I \subset E$ and I generates G . If I is finite, is G finite? Let us construct the free Fischer group generated by I . Let $F(I)$ be the free group generated by I , N the normal subgroup of $F(I)$ generated by the words x^2 , $(s_1 x s_1^{-1} s_2 y s_2^{-1})^3$, $x, y \in I$, $s_1, s_2 \in F(I)$, $G(I) = F(I)/N$, $f: F(I) \rightarrow G(I)$ (the canonical map) and $E(I) \subset G(I)$ the conjugacy class containing $f(I)$. Then $(G(I), E(I))$ is the free Fischer group generated by I and $E(I)$ the free distributive symmetric quasigroup generated by I . By using 2.1 and 2.2 we see that $E(I)$ is finite if I is finite. But unfortunately 2.3 tells us only that $G(I)/Z(G(I))$ is finite.

4. We now proceed to establishing our result. We start with the following combinatorial.

4.1 PROPOSITION. *Let (G, E) be a Fischer group with $\#E = n$. Then*

$$\begin{aligned} \#G &\leq \sum_{k=0}^m \prod_{i=1}^k (n - \frac{1}{2}i(i-1)) \\ &= 1 + n + n(n-1) + \cdots + \prod_{i=1}^m (n - \frac{1}{2}i(i-1)) \end{aligned}$$

where $m = \max\{k \in \mathbb{N} : \frac{1}{2}k(k-1) \leq n\}$.

PROOF. Let W_k denote the irreducible words of length k with letters in E . Then $\#W_0 = 1$, $\#W_1 = n$. We claim that

$$\#W_k \leq \prod_{i=1}^k (n - \frac{1}{2}i(i-1)) \quad \text{if } \frac{1}{2}k(k-1) \leq n, \quad k = 1, 2, \dots$$

Indeed, let $x_1x_2x_3 \cdots x_k \in W_k$. Then for x_1 there are n possible choices, for x_2 there are $n-1$ possible choices ($x_2 \neq x_1$, because otherwise the length of the word could be reduced by 2 since $x^2=1$), for x_3 we have $n-3$ possible choices ($x_3 \neq x_1, x_2$, and $x_1x_2x_1$), for x_4 , $n-6$ choices ($x_4 \neq x_1, x_2, x_3, x_1x_2x_1, x_1x_3x_1, x_2x_3x_2$) etc. Thus the total number of words with irreducible length k is at most $\prod_{i=1}^k (n - \frac{1}{2}i(i-1))$. If $k \geq 2$ this is a pretty liberal estimate, e.g. if $xyx=z \in E$, then $xy=yz=zx$ etc.

4.2 THEOREM. *Every finitely generated Fischer group is finite.*

PROOF. Suppose that the group is generated by a finite set I . It is enough to prove that the free Fischer group $(G(I), E(I))$ is finite. By Proposition 2.2 and Theorem 2.1, $E(I)$ is finite. Therefore $G(I)$ is finite by Proposition 4.1.

5. **Remarks.** Fischer groups were defined in [3] in a more general way involving a general prime p instead of 3. By a theorem due to Fischer (under finiteness assumption) it is obtained that $\#E=3^k$, $\#G=2 \cdot 3^m$ and G is solvable. It would be interesting to know the exact order of G if we know that $\#E=3^k$ and no other relations are introduced except those warranted by the definition of a Fischer group. For example it is easy to see that $\#W_2=3^{k-1}(3^k-1)$.

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