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ON THE INTEGRABILITY OF POWER SERIES

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ABSTRACT. In the present note we obtain a generalization of certain results of Woyczyński [7] concerning equivalence of some statements pertaining to integrability of power series.

1. A nondecreasing continuous real valued function Φ defined on the nonnegative half line and vanishing only at the origin will be called an Orlicz function. Function $\Phi \in OF$ is said to satisfy Δ_α ($\alpha>0$) condition for large u if there are constants $c>0$ and $u_0 \geqq 0$ such that $\Phi(\alpha u) \leqq c\Phi(u)$, $u \geqq u_0$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$$

is called a Young function (YF). Function Φ belongs to YF iff it admits a representation

$$\Phi(u) = \int_0^u \phi(t)\, dt,$$

where $\phi(t)$, $t \geqq 0$, is positive, $\phi(0)=0$, continuous on the right, nondecreasing and $\lim_{t\to\infty} \phi(t) = \infty$.

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [3].

There exist a convex function Λ, $\lambda>1$ *and* $0<\alpha<1$, *such that the inequality* $\Lambda(u) \leqq \Phi^\alpha(u) \leqq \lambda\Lambda(u)$ *holds for all* u.

A sequence $\{a_n\}$ of nonnegative numbers is said to be quasi-monotone if for some $\alpha>0$, $a_{n+1} \leqq a_n(1+\alpha/n)$ ([4], [6]). An equivalent definition of quasi-monotone sequence is that $n^{-\beta}a_n \downarrow 0$ for some $\beta>0$ [5].

Let $L_\Phi(X,\mu)$, where $\Phi \in \Delta_\alpha$, be the Orlicz space, i.e. the set of all complex valued measurable functions f on a measure space (X,μ) such that the modular $\int_X \Phi(|f(x)|)\, d\mu$ is finite. In this paper the Hardy-Orlicz space H_Φ is meant simply to be a closed subset of $L_\Phi((0,2\pi), dx)$ spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int}.$$

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2. Recently Woyczyński [7] proved the following theorem.

THEOREM. *Let* $f(x)=\sum_{k=0}^{\infty} a_k x^k$, $0 \leqq x < 1$. *If*

$$a_n \geqq a_{n+1} \geqq 0 \qquad (n = 0, 1, 2, \cdots),$$

then the following four statements are equivalent:

(2.1) $f(x) \in L_\Phi(0, 1)$;

(2.2) $g(t)=f(e^{it}) \in H_\Phi(0, 2\pi)$;

(2.3) $\{na_n\} \in L_\Phi(N, \nu)$;

(2.4) $\{A_n\} \in L_\Phi(N, \nu)$,

where $\Phi \in \Delta_\alpha \cap M \cap YF$, $d\mu=dx$, *N stands for the set of all positive integers and* ν *is the measure on N concentrating the mass* n^{-2} *at the point* $n \in N$, *and* $A_n=a_0+a_1+a_2+\cdots+a_n$.

The object of this note is to obtain a generalization of the above theorem. Throughout this note, B with or without suffixes, denotes a positive constant, not necessarily the same at each occurrence.

3. We prove the following theorem.

THEOREM. *Let* $\Phi \in \Delta_\alpha \cap M \cap YF$ *and* $f(x)=\sum_{k=0}^{\infty} a_k x^k$, $0 \leqq x < 1$. *If* $\{a_n\}$ *is a quasi-monotone sequence such that* $0 < B_1 \leqq n^\beta a_n \leqq B_2$ *with some* $\beta > 0$, $(n=1, 2, \cdots)$, *and* $0 \leqq \gamma < 1$, *then the following four statements are equivalent*:

(3.1) $(1-x)^{-\gamma}\Phi(|f(x)|) \in L(0, 1)$;

(3.2) $x^{-\gamma}\Phi(|f(e^{ix})|) \in L(0, \pi)$;

(3.3) $\sum_1^\infty n^{\gamma-2}\Phi(na_n) < \infty$;

(3.4) $\sum_1^\infty n^{\gamma-2}\Phi(A_n) < \infty$,

where $A_n=a_0+a_1+a_2+\cdots+a_n$.

REMARK. (3.1) and (3.2)$\Rightarrow$(3.4) if $a_n \geqq 0$.[1]

4. We require the following lemmas for the proof of our theorem.

LEMMA 1 [7]. *Let* $X=R^+$ *and* $d\mu=x^s\,dx$ $(s \leqq 0)$. *If* $\Phi \in M$, *then*

$$\int_X \Phi(F(x)/x)\,d\mu \leqq B \int_X \Phi(f(x))\,d\mu,$$

where

$$F(x) = \int_0^x f(t)\,dt.$$

[1] It is known [1] that (3.4)$\Rightarrow$(3.1) in a much more general situation namely, $0 \leqq \gamma < 2$, Φ increasing, positive and convex and with no restriction on a_n.

LEMMA 2 [7]. *Let $\Phi \in \Delta_\alpha \cap YF$, $X=R^+$ and $d\mu=x^s\,dx$ $(s<-1)$. If $f(x)$ is a nonnegative function and $xf(x) \in L_\Phi(X, \mu)$, then $F(x) \in L_\Phi$, where $F(x)=\int_0^x f(t)\,dt$.*

LEMMA 3 [2].[2] *Let $\{a_n\}$ be positive and tend to zero and $\{n^{-k}a_n\}$ be monotonically decreasing for some nonnegative k, then*

$$\sum_{j=n}^{\infty} |\Delta a_j| < B \sum_{j=n}^{\infty} a_j/j + a_n,$$

where B is some positive constant.

5. **Proof of the theorem.** We shall prove the following implications: (3.1)$\Leftrightarrow$(3.4); (3.4)$\Rightarrow$(3.2), (3.2)$\Rightarrow$(3.3), (3.3)$\Rightarrow$(3.4).

PROOF OF (3.1)$\Rightarrow$(3.4). We write $(1-x)=y$, then by virtue of the fact that $(1-1/n)^n$ $(n=1, 2, \cdots)$ is an increasing sequence, we have for $1/(n+1) \leqq y \leqq 1/n$, $n \geqq 2$,

$$f(1-y) \geqq \sum_{k=0}^{n} a_k(1-y)^k \geqq (1-1/n)^n \sum_{k=0}^{n} a_k \geqq \tfrac{1}{4} A_n.$$

Thus we get

$$f(1-y) \geqq BA_n \quad \text{for } 1/(n+1) \leqq y \leqq 1/n \ (n=2, 3, \cdots).$$

Now

$$\begin{aligned}
\sum_1^\infty n^{\gamma-2}\Phi(A_n) &\leqq B \sum_1^\infty \int_n^{n+1} t^{\gamma-2}\Phi(A_{[t]})\,dt \\
&= B \sum_1^\infty \int_{1/(n+1)}^{1/n} u^{-\gamma}\Phi(A_{[1/u]})\,du \\
&= B \int_{1/2}^{1} u^{-\gamma}\Phi(A_{[1/u]})\,du \\
&\quad + B \sum_{n=2}^{\infty} \int_{1/(n+1)}^{1/n} u^{-\gamma}\Phi(A_n)\,du \\
&\leqq B + B\int_0^{1/2} u^{-\gamma}\Phi(f(1-u))\,du \\
&\leqq B + B\int_0^1 (1-x)^{-\gamma}\Phi(f(x))\,dx < \infty.
\end{aligned}$$

[2] In Lemma 3 Askey has assumed that k should be a positive integer.

PROOF OF (3.4)$\Rightarrow$(3.1).

$$
\begin{aligned}
\int_0^1 (1-x)^{-\gamma}\Phi(f(x))\,dx &= \sum_{n=2}^{\infty}\int_{1-1/(n-1)}^{1-1/n}(1-x)^{-\gamma}\Phi(f(x))\,dx \\
&= \sum_{n=2}^{\infty}\int_{1/n}^{1/(n-1)} x^{-\gamma}\Phi\left(\sum_{k=0}^{\infty} a_k(1-x)^k\right)dx \\
&\leqq \sum_{n=2}^{\infty}\int_{1/n}^{1/(n-1)} x^{-\gamma}\Phi\left(\sum_{k=0}^{\infty} a_k(1-1/n)^k\right)dx \\
&\leqq B\sum_{n=2}^{\infty} n^{\gamma-2}\Phi\left(\sum_{k=0}^{\infty}\sum_{j=nk}^{n(k+1)} a_j(1-1/n)^j\right) \\
&\leqq B\sum_{n=2}^{\infty} n^{\gamma-2}\Phi\left(\sum_{k=0}^{\infty} e^{-k}\left(\sum_{j=0}^{n} a_j + \sum_{nk}^{n(k+1)} a_j\right)\right) \\
&\leqq B\sum_{n=2}^{\infty} n^{\gamma-2}\Phi\left(\sum_{k=0}^{\infty} e^{-k}(A_n + B_2 n^{1-\beta}(k+1))\right) \\
&\leqq B\sum_{n=2}^{\infty} n^{\gamma-2}\Phi\left(\sum_{k=0}^{\infty} B(k+2)e^{-k}A_n\right) \\
&\leqq B\sum_{n=2}^{\infty} n^{\gamma-2}\Phi(A_n) < \infty.
\end{aligned}
$$

PROOF OF (3.4)$\Rightarrow$(3.2). We shall prove that $x^{-\gamma}\Phi(|\mathrm{Re}\, f(e^{ix})|)$ and $x^{-\gamma}\Phi(|\mathrm{Im}\, f(e^{ix})|)$ are both in $L(0,\pi)$.

$$
\begin{aligned}
|\mathrm{Re}\, f(e^{ix})| &= \left|\sum_{k=1}^{\infty} a_k \cos kx\right| \leqq \sum_{k=1}^{n} a_k + \left|\sum_{k=n+1}^{\infty} a_k\cos kx\right| \\
&\leqq A_n + \left|\sum_{n+1}^{\infty} a_k D_k(x) - a_n D_n(x)\right| \\
&\leqq A_n + \quad \left(\sum_{k=n}^{\infty}|\Delta a_k| + a_n\right), \quad \pi/(n+1) \leqq x \leqq \pi/n \\
&\leqq A_n + Bn\left(\sum_{k=n}^{\infty} a_k/k + 2a_n\right), \quad \text{by virtue of Lemma 3} \\
&\leqq A_n + B_1\left(n\sum_{k=n}^{\infty}\frac{1}{k^{1+\beta}} + 2n^{1-\beta}\right) \\
&\leqq A_n + B_1 n^{1-\beta} \leqq B_2 A_n.
\end{aligned}
$$

Now

$$
\begin{aligned}
\int_0^{\pi} x^{-\gamma}\Phi(|\mathrm{Re}\, f(e^{ix})|)\,dx &= \sum_{1}^{\infty}\int_{\pi/(n+1)}^{\pi/n} x^{-\gamma}\Phi(|\mathrm{Re}\, f(e^{ix})|)\,dx \\
&\leqq \sum_{1}^{\infty} n^{\gamma-2}\Phi(B_2 A_n) < \infty.
\end{aligned}
$$

Almost the same proof remains valid for

$$x^{-\gamma}\Phi(|\operatorname{Im} f(e^{ix})|) \in L(0, \pi)$$

and so

$$x^{-\gamma}\Phi(|f(e^{ix})|) \in L(0, \pi),$$

whenever

$$\sum_{n=1}^{\infty} n^{\gamma-2}\Phi(A_n) < \infty.$$

PROOF OF (3.2)$\Rightarrow$(3.3). Let us write $r(t)=\operatorname{Re} f(e^{it})$,

$$R(t) = \int_0^t r(x)\,dx, \qquad R_1(t) = \int_0^t R(x)\,dx.$$

Then

$$\begin{aligned} R_1(t) &= \sum_{j=1}^{\infty} a_j j^{-2}(1-\cos jt) \geqq \sum_{1}^{n} a_j j^{-2}(1-\cos jt) \\ &= 2\sum_{1}^{n} j^{-2} a_j \frac{\sin^2 jt}{2} \\ &\geqq 2\sum_{1}^{n} j^{-2} a_j \frac{4}{\pi^2}\frac{j^2t^2}{4}, \qquad \pi/(n+1) \leqq t \leqq \pi/n \\ &\geqq Bt^2 \sum_{1}^{n} a_j \geqq Bt^2 \cdot na_n. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\gamma-2}\Phi(na_n) &\leqq \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} x^{-\gamma}\Phi\left(\frac{R_1(x)}{x^2}\right) dx \\ &\leqq \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{R_1(x)}{x^2}\right) dx \leqq \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{1}{x^2}\int_0^x |R(t)|\,dt\right) dx \\ &\leqq \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{1}{x}\int_0^x \frac{|R(t)|}{t}\,dt\right) dx \leqq B\int_0^{\pi} x^{-\gamma}\Phi\left(\frac{|R(x)|}{x}\right) dx \\ &\leqq B\int_0^{\pi} x^{-\gamma}\Phi(|r(x)|)\,dx \leqq B\int_0^{\pi} x^{-\gamma}\Phi(|f(e^{ix})|)\,dx < \infty, \end{aligned}$$

by Lemma 1 and by virtue of the hypothesis.

PROOF OF (3.3)$\Rightarrow$(3.4). Let $a(x)$ be the function equal to a_n if $n-1 \leqq x < n$, $n=1, 2, \cdots$, and let $A(x)=\int_0^x a(t)\,dt$. The assumption $\sum n^{\gamma-2}\Phi(na_n)$ implies that $t^{\gamma-2}\Phi(ta(t))$ is integrable on the positive half line, and by virtue of Lemma 2 ($s=\gamma-2<-1$)$t^{\gamma-2}\Phi(A(t))$ is integrable as well. But this is equivalent to the convergence of the series $\sum n^{\gamma-2}\Phi(A_n)$. Hence (3.3)$\Rightarrow$(3.4).

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REFERENCES

1. R. Askey and S. Karlin, *Some elementary integrability theorems for special transforms*, J. Analyse Math. **23** (1970), 27–38. MR **42** #4971.

2. R. Askey and S. Wainger, *Integrability theorems for Fourier series*, Duke Math. J. **33** (1966), 223–228. MR **33** #487.

3. H. P. Mulholland, *On the generalization of Hardy's inequality*, J. London Math. Soc. **7** (1932), 208–214.

4. S. M. Shah, *A note on quasi-monotone series*, Math. Student **15** (1947), 19–24. MR **10**, 446.

5. ———, *Trigonometric series with quasi-monotone coefficients*, Proc. Amer. Math. Soc. **13** (1962), 266–273. MR **27** #4014.

6. O. Szász, *Quasi-monotone series*, Amer. J. Math. **70** (1948), 203–206. MR **9**, 278.

7. W. A. Woyczyński, *Positive coefficient elements of Hardy-Orlicz spaces*, Colloq. Math. **21** (1970), 103–110. MR **41** #2371.

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