

## ON THE INTEGRABILITY OF POWER SERIES

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ABSTRACT. In the present note we obtain a generalization of certain results of Woyczyński [7] concerning equivalence of some statements pertaining to integrability of power series.

1. A nondecreasing continuous real valued function  $\Phi$  defined on the nonnegative half line and vanishing only at the origin will be called an Orlicz function. Function  $\Phi \in OF$  is said to satisfy  $\Delta_\alpha$  ( $\alpha > 0$ ) condition for large  $u$  if there are constants  $c > 0$  and  $u_0 \geq 0$  such that  $\Phi(\alpha u) \leq c\Phi(u)$ ,  $u \geq u_0$ . A convex Orlicz function  $\Phi$  satisfying the conditions

$$\lim_{u \rightarrow 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$$

is called a Young function ( $YF$ ). Function  $\Phi$  belongs to  $YF$  iff it admits a representation

$$\Phi(u) = \int_0^u \phi(t) dt,$$

where  $\phi(t)$ ,  $t \geq 0$ , is positive,  $\phi(0) = 0$ , continuous on the right, non-decreasing and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ .

We denote by  $M$  the class of Orlicz functions  $\Phi$  which satisfy the following condition of Mulholland [3].

*There exist a convex function  $\Lambda$ ,  $\lambda > 1$  and  $0 < \alpha < 1$ , such that the inequality  $\Lambda(u) \leq \Phi^\alpha(u) \leq \lambda \Lambda(u)$  holds for all  $u$ .*

A sequence  $\{a_n\}$  of nonnegative numbers is said to be quasi-monotone if for some  $\alpha > 0$ ,  $a_{n+1} \leq a_n(1 + \alpha/n)$  ([4], [6]). An equivalent definition of quasi-monotone sequence is that  $n^{-\beta} a_n \downarrow 0$  for some  $\beta > 0$  [5].

Let  $L_\Phi(X, \mu)$ , where  $\Phi \in \Delta_\alpha$ , be the Orlicz space, i.e. the set of all complex valued measurable functions  $f$  on a measure space  $(X, \mu)$  such that the modular  $\int_X \Phi(|f(x)|) d\mu$  is finite. In this paper the Hardy-Orlicz space  $H_\Phi$  is meant simply to be a closed subset of  $L_\Phi((0, 2\pi), dx)$  spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^N a_n e^{int}.$$

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2. Recently Woyczyński [7] proved the following theorem.

**THEOREM.** Let  $f(x) = \sum_{k=0}^{\infty} a_k x^k$ ,  $0 \leq x < 1$ . If

$$a_n \geq a_{n+1} \geq 0 \quad (n = 0, 1, 2, \dots),$$

then the following four statements are equivalent:

- (2.1)  $f(x) \in L_{\Phi}(0, 1)$ ;
- (2.2)  $g(t) = f(e^{it}) \in H_{\Phi}(0, 2\pi)$ ;
- (2.3)  $\{na_n\} \in L_{\Phi}(N, \nu)$ ;
- (2.4)  $\{A_n\} \in L_{\Phi}(N, \nu)$ ,

where  $\Phi \in \Delta_{\alpha} \cap M \cap YF$ ,  $d\mu = dx$ ,  $N$  stands for the set of all positive integers and  $\nu$  is the measure on  $N$  concentrating the mass  $n^{-2}$  at the point  $n \in N$ , and  $A_n = a_0 + a_1 + a_2 + \dots + a_n$ .

The object of this note is to obtain a generalization of the above theorem. Throughout this note,  $B$  with or without suffixes, denotes a positive constant, not necessarily the same at each occurrence.

3. We prove the following theorem.

**THEOREM.** Let  $\Phi \in \Delta_{\alpha} \cap M \cap YF$  and  $f(x) = \sum_{k=0}^{\infty} a_k x^k$ ,  $0 \leq x < 1$ . If  $\{a_n\}$  is a quasi-monotone sequence such that  $0 < B_1 \leq n^{\beta} a_n \leq B_2$  with some  $\beta > 0$ , ( $n = 1, 2, \dots$ ), and  $0 \leq \gamma < 1$ , then the following four statements are equivalent:

- (3.1)  $(1-x)^{-\gamma} \Phi(|f(x)|) \in L(0, 1)$ ;
- (3.2)  $x^{-\gamma} \Phi(|f(e^{ix})|) \in L(0, \pi)$ ;
- (3.3)  $\sum_1^{\infty} n^{\gamma-2} \Phi(na_n) < \infty$ ;
- (3.4)  $\sum_1^{\infty} n^{\gamma-2} \Phi(A_n) < \infty$ ,

where  $A_n = a_0 + a_1 + a_2 + \dots + a_n$ .

**REMARK.** (3.1) and (3.2)  $\Rightarrow$  (3.4) if  $a_n \geq 0$ .<sup>1</sup>

4. We require the following lemmas for the proof of our theorem.

**LEMMA 1** [7]. Let  $X = R^+$  and  $d\mu = x^s dx$  ( $s \leq 0$ ). If  $\Phi \in M$ , then

$$\int_X \Phi(F(x)/x) d\mu \leq B \int_X \Phi(f(x)) d\mu,$$

where

$$F(x) = \int_0^x f(t) dt.$$

<sup>1</sup> It is known [1] that (3.4)  $\Rightarrow$  (3.1) in a much more general situation namely,  $0 \leq \gamma < 2$ ,  $\Phi$  increasing, positive and convex and with no restriction on  $a_n$ .

LEMMA 2 [7]. Let  $\Phi \in \Delta_\alpha \cap YF$ ,  $X = R^+$  and  $d\mu = x^s dx$  ( $s < -1$ ). If  $f(x)$  is a nonnegative function and  $xf(x) \in L_\Phi(X, \mu)$ , then  $F(x) \in L_\Phi$ , where  $F(x) = \int_0^x f(t) dt$ .

LEMMA 3 [2].<sup>2</sup> Let  $\{a_n\}$  be positive and tend to zero and  $\{n^{-k}a_n\}$  be monotonically decreasing for some nonnegative  $k$ , then

$$\sum_{j=n}^{\infty} |\Delta a_j| < B \sum_{j=n}^{\infty} a_j/j + a_n,$$

where  $B$  is some positive constant.

5. **Proof of the theorem.** We shall prove the following implications: (3.1)  $\Leftrightarrow$  (3.4); (3.4)  $\Rightarrow$  (3.2), (3.2)  $\Rightarrow$  (3.3), (3.3)  $\Rightarrow$  (3.4).

PROOF OF (3.1)  $\Rightarrow$  (3.4). We write  $(1-x)=y$ , then by virtue of the fact that  $(1-1/n)^n$  ( $n=1, 2, \dots$ ) is an increasing sequence, we have for  $1/(n+1) \leq y \leq 1/n$ ,  $n \geq 2$ ,

$$f(1-y) \geq \sum_{k=0}^n a_k(1-y)^k \geq (1-1/n)^n \sum_{k=0}^n a_k \geq \frac{1}{4}A_n.$$

Thus we get

$$f(1-y) \geq BA_n \quad \text{for } 1/(n+1) \leq y \leq 1/n \quad (n = 2, 3, \dots).$$

Now

$$\begin{aligned} \sum_1^{\infty} n^{\gamma-2} \Phi(A_n) &\leq B \sum_1^{\infty} \int_n^{n+1} t^{\gamma-2} \Phi(A_{[t]}) dt \\ &= B \sum_1^{\infty} \int_{1/(n+1)}^{1/n} u^{-\gamma} \Phi(A_{[1/u]}) du \\ &= B \int_{1/2}^1 u^{-\gamma} \Phi(A_{[1/u]}) du \\ &\quad + B \sum_{n=2}^{\infty} \int_{1/(n+1)}^{1/n} u^{-\gamma} \Phi(A_n) du \\ &\leq B + B \int_0^{1/2} u^{-\gamma} \Phi(f(1-u)) du \\ &\leq B + B \int_0^1 (1-x)^{-\gamma} \Phi(f(x)) dx < \infty. \end{aligned}$$

<sup>2</sup> In Lemma 3 Askey has assumed that  $k$  should be a positive integer.

PROOF OF (3.4)  $\Rightarrow$  (3.1).

$$\begin{aligned}
 \int_0^1 (1-x)^{-\gamma} \Phi(f(x)) dx &= \sum_{n=2}^{\infty} \int_{1-1/(n-1)}^{1-1/n} (1-x)^{-\gamma} \Phi(f(x)) dx \\
 &= \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} x^{-\gamma} \Phi\left(\sum_{k=0}^{\infty} a_k (1-x)^k\right) dx \\
 &\leq \sum_{n=2}^{\infty} \int_{1/n}^{1/(n-1)} x^{-\gamma} \Phi\left(\sum_{k=0}^{\infty} a_k (1-1/n)^k\right) dx \\
 &\leq B \sum_{n=2}^{\infty} n^{\gamma-2} \Phi\left(\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j (1-1/n)^j\right) \\
 &\leq B \sum_{n=2}^{\infty} n^{\gamma-2} \Phi\left(\sum_{k=0}^{\infty} e^{-k} \left(\sum_{j=0}^n a_j + \sum_{nk}^{n(k+1)} a_j\right)\right) \\
 &\leq B \sum_{n=2}^{\infty} n^{\gamma-2} \Phi\left(\sum_{k=0}^{\infty} e^{-k} (A_n + B_2 n^{1-\beta} (k+1))\right) \\
 &\leq B \sum_{n=2}^{\infty} n^{\gamma-2} \Phi\left(\sum_{k=0}^{\infty} B(k+2) e^{-k} A_n\right) \\
 &\leq B \sum_{n=2}^{\infty} n^{\gamma-2} \Phi(A_n) < \infty.
 \end{aligned}$$

PROOF OF (3.4)  $\Rightarrow$  (3.2). We shall prove that  $x^{-\gamma} \Phi(|\operatorname{Re} f(e^{ix})|)$  and  $x^{-\gamma} \Phi(|\operatorname{Im} f(e^{ix})|)$  are both in  $L(0, \pi)$ .

$$\begin{aligned}
 |\operatorname{Re} f(e^{ix})| &= \left| \sum_{k=1}^{\infty} a_k \cos kx \right| \leq \sum_{k=1}^n a_k + \left| \sum_{k=n+1}^{\infty} a_k \cos kx \right| \\
 &\leq A_n + \left| \sum_{k=n+1}^{\infty} a_k D_k(x) - a_n D_n(x) \right| \\
 &\leq A_n + \left( \sum_{k=n}^{\infty} |\Delta a_k| + a_n \right), \quad \pi/(n+1) \leq x \leq \pi/n \\
 &\leq A_n + Bn \left( \sum_{k=n}^{\infty} a_k/k + 2a_n \right), \quad \text{by virtue of Lemma 3} \\
 &\leq A_n + B_1 \left( n \sum_{k=n}^{\infty} \frac{1}{k^{1+\beta}} + 2n^{1-\beta} \right) \\
 &\leq A_n + B_1 n^{1-\beta} \leq B_2 A_n.
 \end{aligned}$$

Now

$$\begin{aligned}
 \int_0^{\pi} x^{-\gamma} \Phi(|\operatorname{Re} f(e^{ix})|) dx &= \sum_1^{\infty} \int_{\pi/(n+1)}^{\pi/n} x^{-\gamma} \Phi(|\operatorname{Re} f(e^{ix})|) dx \\
 &\leq \sum_1^{\infty} n^{\gamma-2} \Phi(B_2 A_n) < \infty.
 \end{aligned}$$

Almost the same proof remains valid for

$$x^{-\gamma}\Phi(|\operatorname{Im} f(e^{ix})|) \in L(0, \pi)$$

and so

$$x^{-\gamma}\Phi(|f(e^{ix})|) \in L(0, \pi),$$

whenever

$$\sum_{n=1}^{\infty} n^{\gamma-2}\Phi(A_n) < \infty.$$

PROOF OF (3.2) $\Rightarrow$ (3.3). Let us write  $r(t)=\operatorname{Re} f(e^{it})$ ,

$$R(t) = \int_0^t r(x) dx, \quad R_1(t) = \int_0^t R(x) dx.$$

Then

$$\begin{aligned} R_1(t) &= \sum_{j=1}^{\infty} a_j j^{-2}(1 - \cos jt) \geq \sum_1^n a_j j^{-2}(1 - \cos jt) \\ &= 2 \sum_1^n j^{-2} a_j \frac{\sin^2 jt}{2} \\ &\geq 2 \sum_1^n j^{-2} a_j \frac{4}{\pi^2} \frac{j^2 t^2}{4}, \quad \pi/(n+1) \leq t \leq \pi/n \\ &\geq Bt^2 \sum_1^n a_j \geq Bt^2 \cdot na_n. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\gamma-2}\Phi(na_n) &\leq \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} x^{-\gamma}\Phi\left(\frac{R_1(x)}{x^2}\right) dx \\ &\leq \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{R_1(x)}{x^2}\right) dx \leq \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{1}{x^2} \int_0^x |R(t)| dt\right) dx \\ &\leq \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{1}{x} \int_0^x \frac{|R(t)|}{t} dt\right) dx \leq B \int_0^{\pi} x^{-\gamma}\Phi\left(\frac{|R(x)|}{x}\right) dx \\ &\leq B \int_0^{\pi} x^{-\gamma}\Phi(|r(x)|) dx \leq B \int_0^{\pi} x^{-\gamma}\Phi(|f(e^{ix})|) dx < \infty, \end{aligned}$$

by Lemma 1 and by virtue of the hypothesis.

PROOF OF (3.3) $\Rightarrow$ (3.4). Let  $a(x)$  be the function equal to  $a_n$  if  $n-1 \leq x < n$ ,  $n=1, 2, \dots$ , and let  $A(x)=\int_0^x a(t) dt$ . The assumption  $\sum n^{\gamma-2}\Phi(na_n)$  implies that  $t^{\gamma-2}\Phi(ta(t))$  is integrable on the positive half line, and by virtue of Lemma 2 ( $s=\gamma-2 < -1$ )  $t^{\gamma-2}\Phi(A(t))$  is integrable as well. But this is equivalent to the convergence of the series  $\sum n^{\gamma-2}\Phi(A_n)$ . Hence (3.3) $\Rightarrow$ (3.4).

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