

REGULAR AND BAER RINGS

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ABSTRACT. The (von Neumann) regular Baer rings representable as the full ring $E(G)$ of all endomorphisms of an abelian group G are characterized. It is also shown that a countable regular Baer ring is Artinian semisimple.

All the rings that we consider are associative with identity and all the modules are unitary left modules. A ring R is called (von Neumann) regular if each $a \in R$ satisfies $a = axa$ for some $x \in R$. R is strongly regular if each a satisfies $a = xa^2 = a^2x$ for some $x \in R$. A ring R is said to be Baer if every left (equivalently, right) annihilator ideal is generated by an idempotent [3]. The additive group of a ring R is denoted by R^+ . Let G be an abelian group. Then $E(G)$ will denote the ring of all endomorphisms of G , G_t the subgroup of all elements of finite order and, for each prime p , G_p the subgroup of all elements of order a power of p . A subgroup S is pure in G if $nS = S \cap nG$ for every positive integer n . A subring S is pure in R if S^+ is pure in R^+ . If $\{A_i\}$, $i \in I$, are rings or groups, then $\prod_{i \in I} A_i$ ($\bigoplus_{i \in I} A_i$) denotes their direct product (sum). \mathcal{Q} denotes the field of rational numbers and $Z(p)$ the field of integers modulo a prime p .

1. Strongly regular Baer rings. First note that a regular ring is strongly regular if and only if it has no nonzero nilpotent elements or, equivalently, all its idempotents are central. Thus for a strongly regular ring R the lattice $L(R)$ of all principal left ideals of R is isomorphic to the Boolean algebra $B(R)$ of all idempotents in R . Moreover the lattice of all left (=right) ideals of R is isomorphic to the lattice of all ideals of $B(R)$. Hence from a well-known theorem of M. H. Stone we get: A strongly regular ring R is Baer $\Leftrightarrow B(R)$ is complete \Leftrightarrow the maximal ideal space of R is extremely disconnected. Since a complete Boolean algebra cannot be countably infinite, the last remark implies that a countable strongly regular Baer ring is Artinian semisimple. Actually a more general result holds, as indicated below.

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THEOREM 1. *A regular Baer ring of cardinality $< c$, the continuum is Artinian semisimple.*

PROOF. It is enough if we show that R satisfies the A.C.C. for principal left ideals. Suppose $Re_1 \subset Re_2 \subset \dots$ is a strictly increasing chain where $e_n = e_n^2$ for all n . Clearly $e_n e_{n+1} = e_n$ for all n . Define inductively $f_1 = e_1$, $f_{n+1} = f_n + e_{n+1} - e_{n+1} f_n$. It is then easy to check that $Re_n = Rf_n$ and $f_n f_{n+1} = f_{n+1} f_n = f_n$ for all n . Hence, by induction, $f_n f_m = f_m f_n = f_n$ for all $m > n$. Define $g_n = f_n - f_{n+1}$. Then $g_n^2 = g_n$ for all n and $g_m g_n = 0$ if $m \neq n$. Note that $g_n \neq 0$, since $Rf_n = Re_n \neq Re_{n+1} = Rf_{n+1}$. To each nonempty subset S of natural numbers, choose an idempotent h_S in R such that the right annihilator of $\{g_i : i \in S\}$ is $(1 - h_S)R$. Clearly $g_i h_S = g_i$ for all $i \in S$, and $(1 - h_S)g_k = g_k$ (i.e., $h_S g_k = 0$) for all $k \notin S$. Note that $h_S g_k = 0$ implies $g_k h_S g_k = 0$ so that $g_k h_S \neq g_k$. Hence if S and T are distinct subsets of natural numbers, then $h_S \neq h_T$. This contradicts the hypothesis that R has cardinality less than c . Hence the result.

2. Regular and Baer endomorphism rings. In this section we completely characterize those regular Baer rings which appear as the full rings of endomorphisms of abelian groups. These rings turn out to be 'almost' self-injective. Since our work here is a continuation of [6], we freely make use of the results and terminology of [6].

LEMMA 1. *Let S be a pure subgroup of $\prod Z(p)^+$, where p runs over a set of primes, such that $\bigoplus Z(p)^+ \subseteq S \subseteq \prod Z(p)^+$. Then the following are true:*

- (i) *if $\rho : S \rightarrow S$ is a Z -morphism then, for any element $x = \langle \dots, x_p, \dots \rangle$ in S , $x\rho = \langle \dots, x_p \rho_p, \dots \rangle$ where $\rho_p = \rho|Z(p)$;*
- (ii) *every endomorphism of S uniquely extends to an endomorphism of $\prod Z(p)^+$.*

LEMMA 2. *Let P be a set of primes and R a pure subring of $\prod_{p \in P} Z(p)$ containing $\bigoplus Z(p)$ and the identity. Then $R \cong \text{Hom}_Z(R^+, R^+)$.*

PROOF. Since $R \cong \text{Hom}_R(R, R)$, it is enough to show that every Z -endomorphism of R is an R -morphism. But this is immediate from Lemma 1(i) and the fact that, for all $p \in P$, $\rho_p = \rho|Z(p)$ is an R -morphism.

As a corollary we get a description of strongly regular endomorphism rings (see Problems 44 and 50 in [1]).

COROLLARY. *Let R be strongly regular. Then $R = E(G)$ for some abelian group G if and only if either $R \cong \prod F_i$, where the F_i are non-isomorphic prime fields or R is a regular subring with identity of $\prod_{p \in P} Z(p)$ containing $\bigoplus Z(p)$, where P is a set of primes.*

PROOF. $R=E(G)$ regular implies (see [5]) that either $G=D\oplus E$, where D is a direct sum of copies of Q^+ and $E=\bigoplus E_p$, each E_p being a direct sum of copies of $Z(p)^+$ or G is a pure subgroup of $\prod G_p$ and each G_p is a direct sum of copies of $Z(p)^+$. If G contains a summand of the form $A\oplus A$, then it is easy to construct an endomorphism $\rho\neq 0$ such that $\rho^2=0$. Since R has no nonzero nilpotent elements, we get, in the first case, that $D\cong Q^+$ and $E_p\cong Z(p)^+$ and, in the second case, that $G_p=Z(p)^+$. This proves the necessity. The sufficiency follows from Lemma 2 if one notes that a regular subring of $\prod Z(p)$ is pure.

We are now ready to prove the main theorem of this paper.

THEOREM 2. *Let R be a regular Baer ring. Then $R\cong E(G)$ for some abelian group G if and only if R is a ring direct sum, $R=R_1\oplus R_2$, where*

(i) R_2 is left self-injective being isomorphic to $\prod_{i\in N} L_i$ where, for each $i\in N$, $L_i\cong \text{Hom}_{F_i}(V_i, V_i)$ with V_i a vector space of dimension ≥ 2 over a prime field F_i and $F_i\ncong F_j$ if $i\neq j$, and

(ii) R_1 is a (commutative) regular subring containing all the idempotents of the ring $\prod_{p\in T} Z(p)$, where T is a set of primes and no p in T is the characteristic of any of the fields F_i . Moreover, $R_1=0$ if any of the fields F_i has characteristic zero.

PROOF. Suppose $R\cong E(G)$ for some abelian group G . Then, by Remark 4.2 of [6], either (a) $G=D\oplus S$, D torsion-free divisible and $S=\bigoplus S_p$ is elementary, p running over a set of primes, or (b) G is reduced, G_t is elementary and G/G_t is divisible. In the case (a), D and S_p can be considered as vector spaces over the fields Q and $Z(p)$ respectively and their Z -endomorphisms are vector space morphisms. Consequently,

$$R \cong \text{Hom}_Q(D, D) \oplus \prod_p \text{Hom}_{Z(p)}(S_p, S_p)$$

which is in the required form. Consider the case (b). We can write $\bigoplus G_p \subseteq G \subseteq \prod G_p$. Since every endomorphism of G uniquely extends to an endomorphism of $\prod G_p$, we consider $R=E(G)$ as a subring of $E(\prod G_p)\cong \prod E(G_p)$ containing $\bigoplus E(G_p)$. Let T be the set of all primes p for which G_p is cyclic. Define $S_1=\bigoplus_{p\in T} G_p$ and $S_2=\bigoplus_{p\notin T} G_p$. If A_i ($i=1, 2$) is the closure of S_i under the n -adic topology of G , then, by Theorem 3.3 of [6], $G=A_1\oplus A_2$. Clearly the A_i are invariant under all the endomorphisms of G so that $R\cong E(A_1)\oplus E(A_2)$.

Let $R_1=E(A_1)$. Since $(A_1)_p=Z(p)^+$, R_1 is isomorphic to a subring of $\prod_{p\in T} Z(p)$ so that R_1 is commutative. Since R_1 is Baer, by Theorem 3.3 and Remark 3.5 of [6], R_1 contains all the idempotents of $\prod_{p\in T} Z(p)$.

Let $R_2=E(A_2)$. Now R_2 is regular and Baer and hence is left continuous, by [6]. Let I be a nonzero two sided ideal of R_2 . We wish to show that I

contains a nonzero nilpotent element. Since $(R_2)_p \cong E(G_p)$ and $\bigoplus (R_2)_p \subseteq R_2 \subseteq \prod (R_2)_p$, and since each $(R_2)_p$ is an elementary abelian p -group, I contains a nonzero ρ satisfying $p\rho=0$ for some prime p . Then $\text{Im } \rho \subset$ the elementary abelian p -group G_p and, composing ρ with a projection of G_p onto a cyclic summand of G_p , get a $\lambda \in I$ with $\text{Im } \lambda$ a cyclic summand, say $G = \text{Im } \lambda \oplus C$. If $\ker \lambda \cap G_p = 0$, then G_p would isomorphically map into $\text{Im } \lambda$ and hence G_p is cyclic, a contradiction. Thus $\ker \lambda \cap G_p \neq 0$. Let μ be an endomorphism of G given by $\mu|C=0$ and $(\mu|_{\text{Im } \lambda})$ a monomorphism from $\text{Im } \lambda$ to $(G_p \cap \ker \lambda)$. Then $0 \neq \lambda\mu \in I$ and $(\lambda\mu)^2=0$. Thus every nonzero two sided ideal of the left continuous regular ring R_2 contains a nonzero nilpotent element. By Theorem 3 of [7], R_2 is therefore left self-injective. Then Theorem 4.6 of [6] implies that $R_2 \cong \prod_{i \in N} L_i$ with $L_i \cong \text{Hom}_{F_i}(V_i, V_i)$, where the F_i 's are prime fields with nonzero characteristic and $F_i \not\cong F_j$ if $i \neq j$.

Thus in both cases, $R=R_1 \oplus R_2$, where the R_i 's have the required form. Moreover, $R_1 \neq 0$ only in the case (b) wherein none of the F_i has characteristic zero.

Conversely, let $R=R_1 \oplus R_2$ be in the required form. Since the F_i are prime fields, $\text{Hom}_{F_i}(V_i, V_i) = \text{Hom}_{\mathbb{Z}}(V_i, V_i)$ and since the F_i are non-isomorphic, $R_2 \cong E(H)$ where $H = \bigoplus V_i^+$. Now by Lemma 2, $R_1 = E(R_1^+)$. If $R_1 \neq 0$, then each F_i has, by hypothesis, characteristic $p_i > 0$ so that, in view of our assumption on the set T ,

$$\text{Hom}_{\mathbb{Z}}(R_1^+, H) = 0 = \text{Hom}_{\mathbb{Z}}(H, R_1^+).$$

Then $R = E(G)$, where $G = H \oplus R_1^+$.

REMARK. If R is regular Baer and $R \cong E(G)$, then R need not be self-injective. This is equivalent to saying that the ring R_1 of Theorem 2 need not be self-injective. To see this, let P be the set of all prime integers > 0 and let A be the subring of $\prod_{p \in P} \mathbb{Z}(p)$ containing $\bigoplus \mathbb{Z}(p)$ and the identity such that $A/(\bigoplus \mathbb{Z}(p)) \cong \mathbb{Q}$, the field of rational numbers (see [2]). Clearly A is regular. Let B be the subring generated by A and all idempotents of $\prod \mathbb{Z}(p)$. Since $B = \sum Ae$, where e runs over all the idempotents of $\prod \mathbb{Z}(p)$, it is easy to see that B is regular (and Baer). Now $B \neq \prod \mathbb{Z}(p)$, since B does not contain the element $x = \langle \dots, x_p, \dots \rangle$, where $x_p^2 \equiv -1 \pmod{p}$, if p is of the form $4n+1$ and $x_p = 0$, otherwise. [Note that, to each $y \in A$, there exists a rational number s/t with $(s, t) = 1$ and $ty_p \equiv s \pmod{p}$ for all except a finite number of primes $p \in P$.] Since B is distinct from its injective envelope $\prod \mathbb{Z}(p)$, B is not self-injective.

By remark 4.3 of [6], we have the following

COROLLARY. *Let G be an abelian group. Then $E(G)$ is regular Baer if and only if either $G = D \oplus E$, where $D \neq 0$ is torsion-free divisible and*

E is elementary or G is reduced and $G = A \oplus B$, where

(i) A is a fully invariant subgroup of $\prod_{p \in P} E_p$ with E_p being an elementary p -group and P a set of primes, and

(ii) B is a pure subgroup of $\prod_{p \in P'} Z(p)^+$ with the property that every endomorphic image and also each subgroup closed under the n -adic topology of B is a direct summand of B , where P' is another set of primes with $P \cap P' = \emptyset$.

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