

SMALL NEIGHBORHOODS OF THE IDENTITY OF A REAL NILPOTENT GROUP¹

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ABSTRACT. It is shown that if G is a real nilpotent group of type D, then for every neighborhood U of the identity in G there is a discrete cocompact subgroup Γ_U of G such that for every $\varphi \in \text{Aut}(G)$, $\varphi\Gamma_U$ and U have more elements in common than just the identity.

This result is exactly the opposite of what is true when G is a semisimple Lie group.

1. Introduction. A Lie group G shall mean a connected Lie group. \hat{G} will denote its Lie algebra and $\text{Aut}(G)$ its group of continuous automorphisms with the C^0 topology. Call G a real nilpotent group if it is a simply connected, nilpotent, real Lie group.

We shall consider discrete cocompact subgroups Γ of G so that G/Γ has a finite measure invariant under the action of G . Let $\Phi(G)$ denote the totality of such Γ 's in G . Let μ be a Haar measure on G . A Borel set P in G is a Γ -packing if $P \cap P\gamma = \emptyset$ for every $\gamma \in \Gamma$, $\gamma \neq e$. $v(\Gamma) = \mu(G/\Gamma)$ shall be called the *volume* of Γ .

Suppose E is a μ -measurable subset of G and $\alpha \in \text{Aut}(G)$. Then $\mu(\alpha(E)) = \Delta(\alpha)\mu(E)$, where $\Delta: \text{Aut}(G) \rightarrow \mathbf{R}_{>0}$ is a homomorphism into the multiplicative group of the positive reals. If G is a real nilpotent group, then $\Delta(\alpha) = |\det \alpha|$, since we can identify G with its Lie algebra \hat{G} , and $\text{Aut}(G)$ with $\text{Aut}(\hat{G})$. Call G *totally unimodular* if the image of Δ is $\{1\}$.

DEFINITION 1. Let \hat{G} be a nilpotent Lie algebra over k , $\text{char } k = 0$. \hat{G} will be called a *Lie algebra of type D* if every derivation of \hat{G} is nilpotent. Call the real nilpotent group $\exp(\hat{G}) = G$ a group of type D. [See [2] for an example.]

The following is known.

(1.1) *If \hat{G} is a real Lie algebra of type D, then $G = \exp(\hat{G})$ is totally unimodular.*

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DEFINITION 2. A nilpotent Lie group G has an expanding automorphism if there exists an automorphism α of G such that with respect to some basis, α has the form

$$\alpha = \text{diag}(a_1, a_2, \dots, a_n), \quad \text{with each } |a_i| > 1.$$

DEFINITION 3. A Lie group G has a KMN if there is a nbhd U of the identity e of G such that for each Γ in $\Phi(G)$ there exists a $\varphi \in \text{Aut}(G)$ such that $\varphi\Gamma \cap U = \{e\}$. Call U a KMN of G .

In [3] it was shown that a semisimple Lie group without compact factors has a KMN and the automorphisms φ are all inner automorphisms. We shall show that for some real nilpotent groups the opposite phenomenon is true.

2. Main theorem. In what follows G will denote a real nilpotent group with a discrete cocompact subgroup Γ .

Suppose X_1, X_2, \dots, X_n is a basis of \hat{G} . Call $\hat{O}_M = \{\sum t_i X_i | 0 \leq t_i < M\}$ the M -ball at 0 in \hat{G} and $\exp \hat{O}_M$ the M -ball at $e \in G$.

DEFINITION 4. A neighborhood V of e in G is *confined* if $V \subset \exp \hat{O}_M$ for some $M > 0$.

This property of nbhds of e is clearly independent of the chosen basis.

(2.1) *If G has an expanding automorphism then every confined nbhd of e is a KMN.*

PROOF. Let V be a confined nbhd and $\Gamma \in \Phi(G)$. Choose M so that $V \subset \exp \hat{O}_M$. Let $\alpha = \text{diag}(a_1, a_2, \dots, a_n)$ be an expanding automorphism. Now pick $m \in \mathbb{Z}^+$ large enough so that $\alpha^m = \text{diag}(a_1^m, a_2^m, \dots, a_n^m)$ maps $\Gamma - \{e\}$ outside of $\exp \hat{O}_M$. Then $\alpha^m \Gamma \cap V = \{e\}$.

As an immediate corollary we have

(2.2) *If \hat{G} is quasi-cyclic, then every confined nbhd of e is a KMN of G . (See [4] for the definition of quasi-cyclic.)*

Our main theorem is

THEOREM 1. *Suppose G is of type D. For each nbhd U of e in G there exists some Γ_U in $\Phi(G)$ such that, for every $\varphi \in \text{Aut}(G)$, $\varphi(\Gamma_U) \cap U \neq \{e\}$.*

PROOF. Suppose the theorem is false and U is a KMN for G . Choose V , a nbhd of e , so that $VV^{-1} \subset U$. Then V is a $\varphi(\Gamma)$ -packing for each Γ in $\Phi(G)$ and for some $\varphi \in \text{Aut}(G)$, φ dependent on the chosen Γ . Take Γ such that $\Gamma \cap V = \{e\}$. Now for every positive integer n we can produce a Γ_n in $\Phi(G)$, $\Gamma_n \supset \Gamma$, and such that the index of Γ in Γ_n , $[\Gamma_n : \Gamma]$, is greater than n (for instance, by taking a canonical basis element in the

center and shrinking it by an appropriate constant factor). Then $v(\Gamma) = [\Gamma_n : \Gamma]v(\Gamma_n)$, so that $v(\Gamma_n) < v(\Gamma)/n$. Take N large enough so that $v(\Gamma)/n < \mu(V)$. Then $v(\Gamma_N) < \mu(V)$. Since U is a KMN, there is a $\varphi \in \text{Aut}(G)$ with $\varphi(\Gamma_N) \cap U = \{e\}$. Since G is type D, by (1.1) we have

$$v(\Gamma_N) = v(\varphi(\Gamma_N)) < \mu(V).$$

But V is a $\varphi(\Gamma_N)$ -packing, so this last relationship is a contradiction.

3. A stronger theorem. Let $\Phi_c(G)$ denote the subset of $\Phi(G)$ consisting of those Γ 's for which $v(\Gamma) \geq c > 0$.

DEFINITION 5. G has a weak KMN if given any $c > 0$ there exists a nbhd W of e in G such that for every Γ in $\Phi_c(G)$ there exists $\varphi \in \text{Aut}(G)$ with $\varphi(\Gamma) \cap W = \{e\}$.

A real nilpotent group may not even have a weak KMN.

THEOREM 2. Suppose G is real nilpotent and totally unimodular. Fix $c > 0$. Then for every nbhd U of e in G , there exists a Γ in $\Phi_c(G)$ with $\varphi(\Gamma) \cap U \neq \{e\}$ for every $\varphi \in \text{Aut}(G)$.

PROOF. Suppose the theorem is false, i.e., assume G has a weak KMN. Then we can show

- Given $c > 0$, there exist $\Gamma_1, \Gamma_2, \dots, \Gamma_s$ in $\Phi(G)$ such that
 (*) for every D in $\Phi(G)$ with $v(D) \leq c$, there exists a $\varphi \in \text{Aut}(G)$ such that $\varphi(D) \cong \Gamma_i$, for some $i = 1, \dots, s$.

Assume (*) for the moment. Using known results it can be shown that, for any given $c > 0$, there are infinitely many nonisomorphic Γ 's in $\Phi(G)$ with $v(\Gamma) \leq c$, a contradiction of (*). So, it is enough to prove (*), given that G has a weak KMN.

PROOF OF (*). Suppose, in fact, that (*) is false. Then we can find a sequence $\{\Gamma_i\}$ such that $v(\Gamma_{i-1}) \leq v(\Gamma_i)$ for all i , and $\lim v(\Gamma_i) = b \leq c$. Set $v(\Gamma_1) = c_1$. Then there exists U_1 a nbhd of e and U_1 is a KMN for $\Phi_{c_1}(G)$. In particular, for each Γ_i in our sequence there is a $\varphi_i \in \text{Aut}(G)$ such that $\varphi_i(\Gamma_i) \cap U_1 = \{e\}$. Take the new sequence $\{\varphi_i(\Gamma_i)\}$ which is uniformly discrete and $v(\varphi_i(\Gamma_i)) \leq c$, for each i , since G is totally unimodular. Now by a theorem of Chabauty [1] there exists a convergent subsequence. The proof now follows the same line of reasoning as H. C. Wang's (8.1) Theorem, in [6].

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