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SMALL NEIGHBORHOODS OF THE IDENTITY OF A REAL NILPOTENT GROUP[1]

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ABSTRACT. It is shown that if G is a real nilpotent group of type D, then for every neighborhood U of the identity in G there is a discrete cocompact subgroup Γ_U of G such that for every $\varphi \in \mathrm{Aut}(G)$, $\varphi\Gamma_U$ and U have more elements in common than just the identity.

This result is exactly the opposite of what is true when G is a semisimple Lie group.

1. **Introduction.** A Lie group G shall mean a connected Lie group. $\hat{G}$ will denote its Lie algebra and $\mathrm{Aut}(G)$ its group of continuous automorphisms with the C-0 topology. Call G a real nilpotent group if it is a simply connected, nilpotent, real Lie group.

We shall consider discrete cocompact subgroups Γ of G so that G/Γ has a finite measure invariant under the action of G. Let $\Phi(G)$ denote the totality of such Γ's in G. Let μ be a Haar measure on G. A Borel set P in G is a Γ-*packing* if $P \cap P\gamma = \varnothing$ for every $\gamma \in \Gamma$, $\gamma \neq e$. $v(\Gamma) = \mu(G/\Gamma)$ shall be called the *volume* of Γ.

Suppose E is a μ-measurable subset of G and $\alpha \in \mathrm{Aut}(G)$. Then $\mu(\alpha(E)) = \Delta(\alpha)\mu(E)$, where $\Delta : \mathrm{Aut}(G) \to \boldsymbol{R}_{>0}$ is a homomorphism into the multiplicative group of the positive reals. If G is a real nilpotent group, then $\Delta(\alpha) = |\det \alpha|$, since we can identify G with its Lie algebra $\hat{G}$, and $\mathrm{Aut}(G)$ with $\mathrm{Aut}(\hat{G})$. Call G *totally unimodular* if the image of Δ is $\{1\}$.

DEFINITION 1. Let $\hat{G}$ be a nilpotent Lie algebra over k, charac $k=0$. $\hat{G}$ will be called a *Lie algebra of type* D if every derivation of $\hat{G}$ is nilpotent. Call the real nilpotent group $\exp(\hat{G}) = G$ a group of type D. [See [2] for an example.]

The following is known.

(1.1) *If $\hat{G}$ is a real Lie algebra of type* D, *then $G = \exp(\hat{G})$ is totally unimodular.*

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[1] These results were contained in the author's doctoral thesis written under the direction of Professor H. C. Wang at Cornell University.

DEFINITION 2. A nilpotent Lie group G has an expanding automorphism if there exists an automorphism α of G such that with respect to some basis, α has the form

$$\alpha = \mathrm{diag}(a_1, a_2, \cdots, a_n), \quad \text{with each } |a_i| > 1.$$

DEFINITION 3. A Lie group G has a KMN if there is a nbhd U of the identity e of G such that for each Γ in $\Phi(G)$ there exists a $\varphi \in \mathrm{Aut}(G)$ such that $\varphi\Gamma \cap U = \{e\}$. Call U a KMN of G.

In [3] it was shown that a semisimple Lie group without compact factors has a KMN and the automorphisms φ are all inner automorphisms. We shall show that for some real nilpotent groups the opposite phenomenon is true.

2. **Main theorem.** In what follows G will denote a real nilpotent group with a discrete cocompact subgroup Γ.

Suppose $X_1, X_2, \cdots, X_n$ is a basis of $\hat{G}$. Call $\hat{O}_M = \{\sum t_i X_i | 0 \leqq t_i < M\}$ the M-ball at 0 in $\hat{G}$ and $\exp \hat{O}_M$ the M-ball at $e \in G$.

DEFINITION 4. A neighborhood V of e in G is *confined* if $V \subset \exp \hat{O}_M$ for some $M > 0$.

This property of nbhds of e is clearly independent of the chosen basis.

(2.1) *If G has an expanding automorphism then every confined nbhd of e is a KMN.*

PROOF. Let V be a confined nbhd and $\Gamma \in \Phi(G)$. Choose M so that $V \subset \exp \hat{O}_M$. Let $\alpha = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ be an expanding automorphism. Now pick $m \in Z^+$ large enough so that $\alpha^m = \mathrm{diag}(a_1^m, a_2^m, \cdots, a_n^m)$ maps $\Gamma - \{e\}$ outside of $\exp \hat{O}_M$. Then $\alpha^m \Gamma \cap V = \{e\}$.

As an immediate corollary we have

(2.2) *If $\hat{G}$ is quasi-cyclic, then every confined nbhd of e is a KMN of G.* (*See* [4] *for the definition of quasi-cyclic.*)

Our main theorem is

THEOREM 1. *Suppose G is of type* D. *For each nbhd U of e in G there exists some Γ_U in $\Phi(G)$ such that, for every $\varphi \in \mathrm{Aut}(G)$, $\varphi(\Gamma_U) \cap U \neq \{e\}$.*

PROOF. Suppose the theorem is false and U is a KMN for G. Choose V, a nbhd of e, so that $VV^{-1} \subset U$. Then V is a $\varphi(\Gamma)$-packing for each Γ in $\Phi(G)$ and for some $\varphi \in \mathrm{Aut}(G)$, φ dependent on the chosen Γ. Take Γ such that $\Gamma \cap V = \{e\}$. Now for every positive integer n we can produce a Γ_n in $\Phi(G)$, $\Gamma_n \supset \Gamma$, and such that the index of Γ in Γ_n, $[\Gamma_n : \Gamma]$, is greater than n (for instance, by taking a canonical basis element in the center and shrinking it by an appropriate constant factor). Then $v(\Gamma)=[\Gamma_n:\Gamma]v(\Gamma_n)$, so that $v(\Gamma_n)<v(\Gamma)/n$. Take N large enough so that $v(\Gamma)/n<\mu(V)$. Then $v(\Gamma_N)<\mu(V)$. Since U is a KMN, there is a $\varphi \in \mathrm{Aut}(G)$ with $\varphi(\Gamma_N)\cap U=\{e\}$. Since G is type D, by (1.1) we have

$$v(\Gamma_N) = v(\varphi(\Gamma_N)) < \mu(V).$$

But V is a $\varphi(\Gamma_N)$-packing, so this last relationship is a contradiction.

3. **A stronger theorem.** Let $\Phi_c(G)$ denote the subset of $\Phi(G)$ consisting of those Γ's for which $v(\Gamma)\geqq c>0$.

DEFINITION 5. G has a weak KMN if given any $c>0$ there exists a nbhd W of e in G such that for every Γ in $\Phi_c(G)$ there exists $\varphi \in \mathrm{Aut}(G)$ with $\varphi(\Gamma)\cap W=\{e\}$.

A real nilpotent group may not even have a weak KMN.

THEOREM 2. *Suppose G is real nilpotent and totally unimodular. Fix $c>0$. Then for every nbhd U of e in G, there exists a Γ in $\Phi_c(G)$ with $\varphi(\Gamma)\cap U\neq\{e\}$ for every $\varphi \in \mathrm{Aut}(G)$.*

PROOF. Suppose the theorem is false, i.e., assume G has a weak KMN. Then we can show

(∗) Given $c>0$, there exist $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ in $\Phi(G)$ such that for every D in $\Phi(G)$ with $v(D)\leqq c$, there exists a $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(D)\simeq\Gamma_i$, for some $i=1, \cdots, s$.

Assume (∗) for the moment. Using known results it can be shown that, for any given $c>0$, there are infinitely many nonisomorphic Γ's in $\Phi(G)$ with $v(\Gamma)\leqq c$, a contradiction of (∗). So, it is enough to prove (∗), given that G has a weak KMN.

PROOF OF (∗). Suppose, in fact, that (∗) is false. Then we can find a sequence $\{\Gamma_i\}$ such that $v(\Gamma_{i-1})\leqq v(\Gamma_i)$ for all i, and $\lim v(\Gamma_i)=b\leqq c$. Set $v(\Gamma_1)=c_1$. Then there exists U_1 a nbhd of e and U_1 is a KMN for $\Phi_{c_i}(G)$. In particular, for each Γ_i in our sequence there is a $\varphi_i \in \mathrm{Aut}(G)$ such that $\varphi_i(\Gamma_i)\cap U_1=\{e\}$. Take the new sequence $\{\varphi_i(\Gamma_i)\}$ which is uniformly discrete and $v(\varphi_i(\Gamma_i))\leqq c$, for each i, since G is totally unimodular. Now by a theorem of Chabauty **[1]** there exists a convergent subsequence. The proof now follows the same line of reasoning as H. C. Wang's (8.1) Theorem, in **[6]**.

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