PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

A NOTE ON THE HOMEOMORPHISM GROUP OF THE RATIONAL NUMBERS

WAYNE R. PARK

ABSTRACT. Let Q be the rational numbers with the usual topology, $H(Q)$ the group of homeomorphisms of Q, γ_c the convergence structure of continuous convergence, and σ the coarsest admissible convergence structure which makes $H(Q)$ a convergence group. A counterexample is constructed to show that if κ is a convergence structure on $H(Q)$ such that $\gamma_c \leqq \kappa \leqq \sigma$, then κ is never principal, hence never topological.

In a previous work by the same author [4], a convergence structure σ on $H(X)$, the group of homeomorphisms of the convergence space X, was developed which is the coarsest of the admissible convergence group structures on $H(X)$. When X is a locally compact topological space, the σ convergence structure becomes a topology, specifically the g-topology [1]. What happens to σ when X is no longer locally compact was left as an open question. Here a simple counterexample describes how badly nontopological the situation is when $X=Q$, the rational numbers with the usual topology. We use the notation as given in [3] and [4].

Let $H(X)$ represent the group of homeomorphisms of the convergence space (X, τ).

DEFINITION. For each $f \in H(X)$ let $\gamma_c f$ consist of all filters $\mathscr{F}$ on $H(X)$ such that

(1) for all $x \in X$ and for all $\Phi \in \tau x$, $\mathscr{F}(\Phi) \in \tau f(x)$.

Here $\mathscr{F}(\Phi)=\omega(\mathscr{F} \times \Phi)$ where $\omega : H(X) \times X \to X$ is the evaluation mapping. The convergence structure γ_c is called the convergence structure of continuous convergence [2].

DEFINITION. For each $f \in H(X)$ let σf consist of all filters $\mathscr{F}$ on $H(X)$ such that $\mathscr{F}$ and $\mathscr{F}^{-1}$ simultaneously belong to $\gamma_c f$.

Here $\mathscr{F}^{-1}$ is the filter with filter base $\{F^{-1} | F \in \mathscr{F}\}$ where $F^{-1} = \{f^{-1} \in H(X) | f \in F\}$.

It is clear that σ is a finer convergence structure than γ_c ($\gamma_c \leqq \sigma$). Moreover for $X=Q$, the rational numbers with the usual topology, σ is strictly finer than γ_c [4]. It is known that γ_c on $H(Q)$ is not a topology [1], and

Presented to the Society, January 19, 1972 under the title *Convergence structures on homeomorphism groups*; received by the editors June 5, 1972 and, in revised form, January 2, 1973.

AMS (MOS) subject classifications (1970). Primary 54A20, 57E05.

moreover is not even a principal convergence structure [**2**]. Here we show the same is true of σ on $H(Q)$, and in fact, true for any convergence structure between γ_c and σ.

THEOREM. *If κ is a convergence structure on $H(Q)$ such that $\gamma_c \leqq \kappa \leqq \sigma$, then κ is not a principal convergence structure.*

PROOF. Let I be the set of irrational numbers. For each $\alpha \in I$, define a sequence of homeomorphisms $f_{n,\alpha}$ in $H(Q)$ by

$$\begin{aligned} f_{n,\alpha}(x) &= -x \quad \text{if } |\alpha| - 1/n < |x| < |\alpha| + 1/n, \\ &= x \quad\ \ \text{otherwise}, \end{aligned}$$

where $x \in Q$, and the integer n is larger than $1/|\alpha|$.

For each $\alpha \in I$, let $F_{N,\alpha} = \{f_{n,\alpha} \in H(Q) | n \geqq N\}$ and let $\mathscr{F}_\alpha$ be the filter generated by the filter base $\{F_{N,\alpha}\}$ where N runs through the integers greater than $1/|\alpha|$. As $f^{-1}_{n,\alpha} = f_{n,\alpha}$, it follows that $\mathscr{F}_\alpha = \mathscr{F}^{-1}_\alpha$ for each $\alpha \in I$. Moreover, $\mathscr{F}_\alpha \in \gamma_c(\mathrm{id})$ where id denotes the identity homeomorphism. Namely, let $q \in Q$ and let Φ denote the neighborhood filter of q with the usual topology (convergence structure) τ. To satisfy condition (1) above, let U be arbitrary in Φ. One can find N sufficiently large and a V in Φ such that $\omega(F_{N,\alpha} \times V) \subseteq U$. Hence $\mathscr{F}_\alpha(\Phi) \geqq \Phi$, so $\mathscr{F}_\alpha(\Phi)$ converges topologically to $\mathrm{id}(q) = q$, that is, $\mathscr{F}_\alpha(\Phi) \in \tau\ \mathrm{id}(q)$.

Now as $\mathscr{F}_\alpha = \mathscr{F}^{-1}_\alpha$ and as $\mathscr{F}_\alpha \in \gamma_c(\mathrm{id})$, it follows that $\mathscr{F}_\alpha \in \sigma(\mathrm{id})$ for each $\alpha \in I$.

But now, if q is any nonzero rational number and Φ its neighborhood filter, $\omega(\bigwedge_\alpha \mathscr{F}_\alpha \times \Phi) = (\bigwedge_\alpha \mathscr{F}_\alpha)(\Phi)$ does not converge to $\mathrm{id}(q) = q$. If it did, for any U in Φ we would need to find a V in Φ and a filter base element $\bigcup_\alpha F_{N_\alpha,\alpha}$ in $\bigwedge_\alpha \mathscr{F}_\alpha$ such that $\omega(\bigcup_\alpha F_{N_\alpha,\alpha} \times V) \subseteq U$. As irrationals lie arbitrarily close to q, this step is impossible.

We have constructed a family of filters $\{\mathscr{F}_\alpha\}$ such that $\mathscr{F}_\alpha \in \sigma(\mathrm{id})$ for each $\alpha \in I$ but such that $\bigwedge_\alpha \mathscr{F}_\alpha \notin \gamma_c(\mathrm{id})$. This shows that κ cannot be a principal convergence structure.

BIBLIOGRAPHY

1. R. F. Arens, *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946), 593–610. MR **8**, 479.

2. C. H. Cook and H. R. Fischer, *On equicontinuity and continuous convergence*, Math. Ann. **159** (1965), 94–104. MR **31** #3995.

3. H. R. Fischer, *Limesräume*, Math. Ann. **137** (1959), 269–303. MR **22** #225.

4. W. R. Park, *Convergence structures on homeomorphism groups*, Math. Ann. **199** (1972), 45–54.

DEPARTMENT OF MATHEMATICS, ST. LAWRENCE UNIVERSITY, CANTON, NEW YORK 13617