A NOTE ON THE HOMEOMORPHISM GROUP
OF THE RATIONAL NUMBERS

WAYNE R. PARK

Abstract. Let $Q$ be the rational numbers with the usual topology, $H(Q)$ the group of homeomorphisms of $Q$, $\gamma_c$ the convergence structure of continuous convergence, and $\sigma$ the coarsest admissible convergence structure which makes $H(Q)$ a convergence group. A counterexample is constructed to show that if $\kappa$ is a convergence structure on $H(Q)$ such that $\gamma_c \leq \kappa \leq \sigma$, then $\kappa$ is never principal, hence never topological.

In a previous work by the same author [4], a convergence structure $\sigma$ on $H(X)$, the group of homeomorphisms of the convergence space $X$, was developed which is the coarsest of the admissible convergence group structures on $H(X)$. When $X$ is a locally compact topological space, the $\sigma$ convergence structure becomes a topology, specifically the $g$-topology [1]. What happens to $\sigma$ when $X$ is no longer locally compact was left as an open question. Here a simple counterexample describes how badly non-topological the situation is when $X=Q$, the rational numbers with the usual topology. We use the notation as given in [3] and [4].

Let $H(X)$ represent the group of homeomorphisms of the convergence space $(X, \tau)$.

Definition. For each $f \in H(X)$ let $\gamma_c f$ consist of all filters $\mathcal{F}$ on $H(X)$ such that

(1) for all $x \in X$ and for all $\Phi \in \tau x$, $\mathcal{F}(\Phi) \in \tau f(x)$.

Here $\mathcal{F}(\Phi) = \omega(\mathcal{F} \times \Phi)$ where $\omega : H(X) \times X \to X$ is the evaluation mapping. The convergence structure $\gamma_c$ is called the convergence structure of continuous convergence [2].

Definition. For each $f \in H(X)$ let $\sigma f$ consist of all filters $\mathcal{F}$ on $H(X)$ such that $\mathcal{F}$ and $\mathcal{F}^{-1}$ simultaneously belong to $\gamma_c f$.

Here $\mathcal{F}^{-1}$ is the filter with filter base $\{F^{-1} \mid F \in \mathcal{F}\}$ where $F^{-1} = \{f^{-1} \in H(X) \mid f \in F\}$.

It is clear that $\sigma$ is a finer convergence structure than $\gamma_c$ ($\gamma_c \leq \sigma$). Moreover for $X=Q$, the rational numbers with the usual topology, $\sigma$ is strictly finer than $\gamma_c$ [4]. It is known that $\gamma_c$ on $H(Q)$ is not a topology [1], and
moreover is not even a principal convergence structure [2]. Here we show the same is true of $\sigma$ on $H(Q)$, and in fact, true for any convergence structure between $\gamma_c$ and $\sigma$.

**Theorem.** If $\kappa$ is a convergence structure on $H(Q)$ such that $\gamma_c \subseteq \kappa \subseteq \sigma$, then $\kappa$ is not a principal convergence structure.

**Proof.** Let $I$ be the set of irrational numbers. For each $a \in I$, define a sequence of homeomorphisms $f_{n,a}$ in $H(Q)$ by

$$f_{n,a}(x) = \begin{cases} 
-x & \text{if } |x| - 1/n < |x| < |x| + 1/n, \\
\frac{x}{a} & \text{otherwise},
\end{cases}$$

where $x \in Q$, and the integer $n$ is larger than $1/|a|$.

For each $a \in I$, let $F_{N,a} = \{f_{n,a} \in H(Q) | n \geq N\}$ and let $\mathcal{F}_a$ be the filter generated by the filter base $\{F_{N,a}\}$ where $N$ runs through the integers greater than $1/|a|$. As $f_{n,a}^{-1} = f_{n,a}$, it follows that $\mathcal{F}_a = \mathcal{F}_a^{-1}$ for each $a \in I$. Moreover, $\mathcal{F}_a \in \gamma_c(id)$, where $id$ denotes the identity homeomorphism. Namely, let $q \in Q$ and let $\Phi$ denote the neighborhood filter of $q$ with the usual topology (convergence structure) $\tau$. To satisfy condition (1) above, let $U$ be arbitrary in $\Phi$. One can find $N$ sufficiently large and a $V$ in $\Phi$ such that $\omega(F_{N,a} \times V) \subseteq U$. Hence $\mathcal{F}_a(\Phi) \subseteq \Phi$, so $\mathcal{F}_a(\Phi)$ converges topologically to $id(q) = q$, that is, $\mathcal{F}_a(\Phi) \subseteq \tau(id(q))$.

Now as $\mathcal{F}_a = \mathcal{F}_a^{-1}$ and as $\mathcal{F}_a \in \gamma_c(id)$, it follows that $\mathcal{F}_a \in \sigma(id)$ for each $a \in I$.

But now, if $q$ is any nonzero rational number and $\Phi$ its neighborhood filter, $\omega(\bigwedge_a \mathcal{F}_a \times \Phi) = (\bigwedge_a \mathcal{F}_a)(\Phi)$ does not converge to $id(q) = q$. If it did, for any $U$ in $\Phi$ we would need to find a $V$ in $\Phi$ and a filter base element $\bigcup_a F_{N,a}$ in $\bigwedge_a \mathcal{F}_a$ such that $\omega(\bigcup_a F_{N,a} \times V) \subseteq U$. As irrationals lie arbitrarily close to $q$, this step is impossible.

We have constructed a family of filters $\{\mathcal{F}_a\}$ such that $\mathcal{F}_a \in \sigma(id)$ for each $a \in I$ but such that $\bigwedge_a \mathcal{F}_a \notin \gamma_c(id)$. This shows that $\kappa$ cannot be a principal convergence structure.

**Bibliography**


Department of Mathematics, St. Lawrence University, Canton, New York 13617