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# A NOTE ON THE HOMEOMORPHISM GROUP OF THE RATIONAL NUMBERS

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ABSTRACT. Let $Q$ be the rational numbers with the usual topology, $H(Q)$ the group of homeomorphisms of $Q$, $\gamma_c$ the convergence structure of continuous convergence, and $\sigma$ the coarsest admissible convergence structure which makes $H(Q)$ a convergence group. A counterexample is constructed to show that if $\kappa$ is a convergence structure on $H(Q)$ such that $\gamma_c \leqq \kappa \leqq \sigma$, then $\kappa$ is never principal, hence never topological.

In a previous work by the same author [**4**], a convergence structure $\sigma$ on $H(X)$, the group of homeomorphisms of the convergence space $X$, was developed which is the coarsest of the admissible convergence group structures on $H(X)$. When $X$ is a locally compact topological space, the $\sigma$ convergence structure becomes a topology, specifically the $g$-topology [**1**]. What happens to $\sigma$ when $X$ is no longer locally compact was left as an open question. Here a simple counterexample describes how badly nontopological the situation is when $X=Q$, the rational numbers with the usual topology. We use the notation as given in [**3**] and [**4**].

Let $H(X)$ represent the group of homeomorphisms of the convergence space $(X, \tau)$.

DEFINITION. For each $f \in H(X)$ let $\gamma_c f$ consist of all filters $\mathscr{F}$ on $H(X)$ such that

(1) for all $x \in X$ and for all $\Phi \in \tau x$, $\mathscr{F}(\Phi) \in \tau f(x)$.

Here $\mathscr{F}(\Phi)=\omega(\mathscr{F} \times \Phi)$ where $\omega: H(X) \times X \to X$ is the evaluation mapping. The convergence structure $\gamma_c$ is called the convergence structure of continuous convergence [**2**].

DEFINITION. For each $f \in H(X)$ let $\sigma f$ consist of all filters $\mathscr{F}$ on $H(X)$ such that $\mathscr{F}$ and $\mathscr{F}^{-1}$ simultaneously belong to $\gamma_c f$.

Here $\mathscr{F}^{-1}$ is the filter with filter base $\{F^{-1} | F \in \mathscr{F}\}$ where $F^{-1}=\{f^{-1} \in H(X) | f \in F\}$.

It is clear that $\sigma$ is a finer convergence structure than $\gamma_c$ ($\gamma_c \leqq \sigma$). Moreover for $X=Q$, the rational numbers with the usual topology, $\sigma$ is strictly finer than $\gamma_c$ [**4**]. It is known that $\gamma_c$ on $H(Q)$ is not a topology [**1**], and

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moreover is not even a principal convergence structure [2]. Here we show the same is true of $\sigma$ on $H(Q)$, and in fact, true for any convergence structure between $\gamma_c$ and $\sigma$.

THEOREM. *If $\kappa$ is a convergence structure on $H(Q)$ such that $\gamma_c \leqq \kappa \leqq \sigma$, then $\kappa$ is not a principal convergence structure.*

PROOF. Let $I$ be the set of irrational numbers. For each $\alpha \in I$, define a sequence of homeomorphisms $f_{n,\alpha}$ in $H(Q)$ by

$$\begin{aligned} f_{n,\alpha}(x) &= -x \quad \text{if } |\alpha| - 1/n < |x| < |\alpha| + 1/n, \\ &= x \quad \text{otherwise,} \end{aligned}$$

where $x \in Q$, and the integer $n$ is larger than $1/|\alpha|$.

For each $\alpha \in I$, let $F_{N,\alpha} = \{f_{n,\alpha} \in H(Q) | n \geqq N\}$ and let $\mathscr{F}_\alpha$ be the filter generated by the filter base $\{F_{N,\alpha}\}$ where $N$ runs through the integers greater than $1/|\alpha|$. As $f_{n,\alpha}^{-1} = f_{n,\alpha}$, it follows that $\mathscr{F}_\alpha = \mathscr{F}_\alpha^{-1}$ for each $\alpha \in I$. Moreover, $\mathscr{F}_\alpha \in \gamma_c(\text{id})$ where id denotes the identity homeomorphism. Namely, let $q \in Q$ and let $\Phi$ denote the neighborhood filter of $q$ with the usual topology (convergence structure) $\tau$. To satisfy condition (1) above, let $U$ be arbitrary in $\Phi$. One can find $N$ sufficiently large and a $V$ in $\Phi$ such that $\omega(F_{N,\alpha} \times V) \subseteq U$. Hence $\mathscr{F}_\alpha(\Phi) \geqq \Phi$, so $\mathscr{F}_\alpha(\Phi)$ converges topologically to $\text{id}(q) = q$, that is, $\mathscr{F}_\alpha(\Phi) \in \tau\, \text{id}(q)$.

Now as $\mathscr{F}_\alpha = \mathscr{F}_\alpha^{-1}$ and as $\mathscr{F}_\alpha \in \gamma_c(\text{id})$, it follows that $\mathscr{F}_\alpha \in \sigma(\text{id})$ for each $\alpha \in I$.

But now, if $q$ is any nonzero rational number and $\Phi$ its neighborhood filter, $\omega(\bigwedge_\alpha \mathscr{F}_\alpha \times \Phi) = (\bigwedge_\alpha \mathscr{F}_\alpha)(\Phi)$ does not converge to $\text{id}(q) = q$. If it did, for any $U$ in $\Phi$ we would need to find a $V$ in $\Phi$ and a filter base element $\bigcup_\alpha F_{N_\alpha,\alpha}$ in $\bigwedge_\alpha \mathscr{F}_\alpha$ such that $\omega(\bigcup_\alpha F_{N_\alpha,\alpha} \times V) \subseteq U$. As irrationals lie arbitrarily close to $q$, this step is impossible.

We have constructed a family of filters $\{\mathscr{F}_\alpha\}$ such that $\mathscr{F}_\alpha \in \sigma(\text{id})$ for each $\alpha \in I$ but such that $\bigwedge_\alpha \mathscr{F}_\alpha \notin \gamma_c(\text{id})$. This shows that $\kappa$ cannot be a principal convergence structure.

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