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A STABILITY THEOREM FOR A REAL ANALYTIC SINGULAR CAUCHY PROBLEM

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ABSTRACT. In this paper we prove the equation $u_{tt}-t^{2p}u_{xx}-a(t)u_x=0, p>0$, with initial conditions $u(x,0)=\alpha(x)$, $u_t(x,0)=\beta(x)$ is well posed provided that $\alpha(x)$ and $\beta(x)$ belong to special classes of real analytic functions. In general this problem is not stable for $p>1$ and $\alpha(x)$ and $\beta(x)$ real analytic functions.

1. **Introduction.** Let $A(R)$ be the class of all real valued functions which are represented by power series expansions on the interval $(-R, R)$. In general the Cauchy problem

$$u_{tt}-t^{2p}u_{xx}-a(t)u_x=0, \qquad p>0, \tag{1}$$

$$u(x,0)=\alpha(x), \qquad u_t(x,0)=\beta(x), \tag{2}$$

is not well posed if $\alpha(x)$ and $\beta(x)$ belong to $A(R)$. In fact for $p>1$ the stability in the uniform metric may be violated (for an example see [**1**]).

In this paper we define special classes $H(R)$ of functions in $A(R)$ and prove a theorem giving a well-posed problem provided $\alpha(x)$ and $\beta(x)$ are restricted to belong to such a class $H(R)$. M. H. Protter [**6**] gave a condition for a more general problem which implies that (1), (2) is well posed if $\lim_{t\to 0+} t^{1-p}a(t)=0$. For further papers on problems of this nature see ([**1**], [**3**], [**4**], [**5**], [**8**]). For general abstract existence and uniqueness theorems see [**2**] and [**3**]. In particular A. B. Nersesjan in [**5**] states a theorem which shows that the Cauchy problem is well posed in the case that z is a complex variable, $\alpha(z)$ and $\beta(z)$ are analytic for $|z|<R$ and solutions are admitted in the class of functions $u(z,t)$ such that, for each fixed t, $u(z,t)$ is analytic for $|z|<R$. We shall see that stability occurs for $p>1$ in the complex variable case, as opposed to the real variable case, because of the availability of the Cauchy estimates. Namely if z is complex and

$$\max_{0\leq|z|\leq\rho}\left|\sum_{n=0}^{\infty}\alpha_n z^n-\sum_{n=0}^{\infty}\phi_n z^n\right|=\delta$$

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then for each n, $|\alpha_n - \beta_n| \leqq \delta/\rho^n$. Of course such estimates are not available in the case that x is a real variable; however, we define a more general estimate of this nature which we call a G estimate and show that in the real analytic case we have stability provided that $\alpha(x)$ and $\beta(x)$ are restricted to a class of functions which have the G estimate property.

2. **Definitions and preliminaries.** Let I be an index set and let $H(R) = \{\alpha_h(x)\}_{h \in I}$ be a class of functions in $A(R)$.

DEFINITION. $H(R)$ will be said to have the G estimate property if the following condition is satisfied: Suppose $0 < \rho < R$ and $\alpha(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ and $\phi(x) = \sum_{n=0}^{\infty} \phi_n x^n$ are any two functions in $H(R)$. Let

$$\delta(\rho) = \max_{0 \leqq |x| \leqq \rho} |\alpha(x) - \phi(x)|;$$

then for every nonnegative integer n,

$$|\alpha_n - \phi_n| \leqq g(n, \rho)\delta(\rho)/\rho^n$$

where $g(n, \rho) \leqq c(\rho) \prod_{i=0}^{N} (n+i)$, $c(\rho)$ positive and N a fixed positive integer.

EXAMPLES. (i) Suppose $\{b_n(h)\}$ is a sequence of nonnegative monotone increasing functions each defined for $h \geqq 0$ and satisfying $b_n(h) \leqq 1$. Define

$$H(1) = \left\{ \sum_{n=0}^{\infty} (-1)^n b_n(h) x^n \right\}_{h \in [0, \infty)},$$

then $H(1)$ has the G estimate property with $g(n, \rho) = 1$. It is easy to show that if $0 \leqq h < k$ and z is a complex variable then

$$\left| \sum_{n=0}^{\infty} (-1)^n (b_n(k) - b_n(h)) z^n \right|$$

assumes its maximum on $|z| = \rho$ at $z = -\rho$. Then the Cauchy estimates for the disc $|z| \leqq \rho$ in the complex plane give the required results for the interval $|x| \leqq \rho$ on the real line.

(ii) A simple example in which $g(n, \rho)$ must be a function of n is

$$H(1) = \{h/(1 + x^2)^2\}_{h \in [0, \infty)}.$$

(iii) If $H(1) = \{\sin hx\}_{h \in [0, \infty)}$ we have a class of functions which can never satisfy the G estimate property.

DEFINITION. Suppose $0 < r < \rho < R$, $0 < \theta$, $0 < X$,

$$X + \theta^{p+1}/(p + 1) = r,$$

and $S=\{(x,t):0\leqq t\leqq\theta,\ |x|\leqq X\}$. The Cauchy problem (1), (2) is said to be G stable on S with respect to the class of functions $H(R)$ if for each $\varepsilon>0$ there exists $\delta>0$ such that whenever $\alpha_1(x)$, $\alpha_2(x)$, $\beta_1(x)$, $\beta_2(x)$, belong to $H(R)$ and $\max_{0\leqq|x|\leqq\rho}|\alpha_1(x)-\alpha_2(x)|<\delta$, $\max_{0\leqq|x|\leqq\rho}|\beta_1(x)-\beta_2(x)|<\delta$, then if $u_i(x,t)$ is the solution of (1), (2) with $\alpha(x)=\alpha_i(x)$, $\beta(x)=\beta_i(x)$, $i=1,2$, it follows that

$$\max_{(x,t)\in S}|u_1(x,t)-u_2(x,t)|<\varepsilon.$$

LEMMA.

$$\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty}g(n+w+v)\frac{(n+w+v)!}{n!\,w!\,v!}\alpha^n\beta^v\frac{\gamma^w}{w!}$$

converges for all γ *if* $|\alpha|+|\beta|<1$.

PROOF. Choose $\delta>0$ such that $|\alpha|+|\beta|+\delta<1$. Then there exists w_0 such that for all $w\geqq w_0$, $|\gamma|^w/w!<\delta^w$. Now for each fixed $w<w_0$ the double series

$$\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}g(n+w+v)\frac{(n+w+v)!}{n!\,v!}\alpha^n\beta^v$$

converges. To prove this we observe that the Appell series

$$\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\frac{(n+w+v)!}{n!v!}(\alpha^n\beta^v)$$

converges for $|\alpha|+|\beta|<1$ and the proof will be unaltered by the term $g(n+w+v)$ since $g(n)\leqq c(\rho)\prod_{i=0}^{N}(n+i)$ (for the proof of the convergence of the Appell series see [7, pp. 210–213]). For $w\geqq w_0$ the triple series is dominated by

$$\sum_{n=0}^{\infty}\sum_{w=w_0}^{\infty}\sum_{v=0}^{\infty}g(n+w+v)\frac{(n+w+v)!}{n!\,w!\,v!}|\alpha|^n\,|\beta|^n\,\delta^w$$

and again, as above, convergence follows since $|\alpha|+|\beta|+\delta<1$ (see Lauricella functions [7, p. 227]).

3. THEOREM. *If the class of functions* $H(R)$ *has the G estimate property then the Cauchy problem* (1), (2) *is G stable on S with respect to* $H(R)$.

PROOF. Suppose $\alpha_1(x)-\alpha_2(x)=\sum_{n=0}^{\infty}\alpha_nx^n$,

$$\beta_1(x)-\beta_2(x)=\sum_{n=0}^{\infty}\beta_nx^n,\qquad 0<r<\rho<R,$$

and $X+\theta^{p+1}/(p+1)=r$. We need only assume that $a(t)$ is continuous.

Integrating (1) twice with respect to t, we obtain

$$u - Ku_{xx} - Lu_x = \alpha(x) + t\beta(x) \tag{3}$$

where the operators K and L are defined by

$$(Kf)(t) = \int_0^t \int_0^r s^{2p} f(s)\, ds\, dr, \qquad (Lf)(t) = \int_0^t \int_0^r a(s) f(s)\, ds\, dr,$$

for $f(t)$ a real valued continuous function defined on an interval $0 \leqq t \leqq \theta$.

We shall denote by $I(w, v)$ the summation over all distinct operators obtained by applying L w times and K v times and define $f_n(t) = \alpha_n + \beta_n t$. Let $\| \ \|$ denote the supremum norm on $C[0, \theta]$. Then by the G estimate property

$$\|f_n\| \leqq |\alpha_n| + |\beta_n|\,\theta \leqq (1 + \theta) g(n) \delta / \rho^n$$

where for convenience we use the abbreviations $g(n) = g(n, \rho)$ and $\delta = \delta(\rho)$. Substitution in (3) shows there will be a solution of the form $u(x, t) = \sum_{n=0}^{\infty} a_n(t) x^n$ if, for each n,

$$a_n(t) = (n+1)(n+2) K a_{n+2}(t) + (n+1) L a_{n+1}(t) + f_n(t). \tag{4}$$

Assuming for the moment that the infinite sum is absolutely convergent we will show that (4) is satisfied if, for all n,

$$a_n(t) = \sum_{w=0}^{\infty} \sum_{v=0}^{\infty} \frac{(n + w + 2v)!}{n!} I(w, v) f_{n+w+2v}(t).$$

It can be seen that

for $w \geqq 1$, $v \geqq 1$, $I(w, v) = KI(w, v-1) + LI(w-1, v)$
for $w = 0$, $v \geqq 1$, $I(0, v) = KI(0, v-1)$
for $w \geqq 1$, $v = 0$, $I(w, 0) = LI(w-1, 0)$.

Then it may be verified directly that (4) is an identity.

It remains to show the infinite sum converges absolutely and to prove G stability on $S = \{(x, t), |x| \leqq X, 0 \leqq t \leqq \theta\}$.

We see that $I(w, v)$ consists of $\binom{w+v}{v}$ operators each obtained by applying L w times and K v times. Further, each operator has norm at most $\theta^{2w+2v(p+1)} \|a\|^w / (2w)! E(v)$ where

$$E(v) = \prod_{j=1}^{v} (2pj + 2j - 1)(2pj + 2j) \geqq (p+1)^{2v} (2v)!.$$

Hence, if we set $T=\sum_{n=0}^{\infty}\|a_n\|X^n$ and denote $\sum_{n=0}\sum_{w=0}\sum_{v=0}$ by $\sum$,

$$T\leqq\sum\binom{w+v}{v}\frac{(n+w+2v)!\,X^n\theta^{2w+2v(p+1)}\,\|a\|^w(1+\theta)g(n+w+2v)\delta}{n!\,(2w)!\,(2v)!\,(p+1)^{2v}\rho^{n+w+2v}}$$

$$\leqq\sum\binom{w+v}{v}\frac{(n+w+v)!}{n!\,w!\,v!}\left(\frac{X}{\rho}\right)^n\left(\frac{\theta^{(p+1)}}{(p+1)\rho}\right)^v\frac{1}{w!}\left(\frac{\theta^2\,\|a\|}{\rho}\right)^w$$

$$\times(1+\theta)g(n+w+v)\delta.$$

Now there exists γ such that $0<\gamma<1$ and

$$X/\rho+(\theta^{p+1}/(p+1)\rho)^{1-\gamma}<1 \tag{5}$$

and there exists $k>0$ such that

$$(\theta^{p+1}/(p+1)\rho)^{\gamma}+k<1. \tag{6}$$

By the lemma and (5)

$$\sum\frac{(n+w+v)!}{n!\,w!\,v!}g(n+w+v)\left(\frac{X}{\rho}\right)^n[(\theta^{p+1}/(p+1)\rho)^{1-\gamma}]^v\frac{(\theta^2\,\|a\|/\rho k)^w}{w!}$$

converges. Hence there exists M such that for all w and v

$$Mk^w\geqq\sum_{n=0}^{\infty}\frac{(n+w+v)!}{n!\,w!\,v!}g(n+w+v)$$

$$\times\left(\frac{X}{\rho}\right)^n[(\theta^{p+1}/(p+1)\rho)^{1-\gamma}]^v\frac{(\theta^2\,\|a\|/\rho)^w}{w!}.$$

Hence

$$T\leqq M(1+\theta)\delta\sum_{w=0}^{\infty}\sum_{v=0}^{\infty}\binom{w+v}{v}[(\theta^{p+1}/(p+1)\rho)^{\gamma}]^v k^w.$$

The latter series converges by (6) and the proof is complete.

References

1. I. S. Berezin, *On Cauchy's problem for linear equations of the second order with initial conditions on a parabolic line*, Mat. Sb. **24** (66) (1949), 301–320; English transl., Amer. Math. Soc. Transl. (1) **4** (1962), 415–439. MR **11**, 112.

2. R. Carroll, *Some degenerate Cauchy problems with operator coefficients*, Pacific J. Math. **13** (1963), 471–485. MR **29** #367.

3. R. Carroll and C. Wang, *On the degenerate Cauchy problem*, Canad. J. Math. **17** (1965), 245–256. MR **36** #489.

4. Chi Min-you, *The Cauchy problem for a class of hyperbolic equations with data on a line of parabolic degeneracy*, Acta. Math. Sinica **8** (1958), 521–529=Chinese Math. **9** (1967), 246–254. MR **21** #5815.

5. A. B. Nersesjan, *The Cauchy problem for degenerating hyperbolic equations of second order*, Dokl. Akad. Nauk SSSR **166** (1966), 1288–1291 = Soviet Math. Dokl. **7** (1966), 278–281. MR **33** #4465.

6. M. H. Protter, *The Cauchy problem for a hyperbolic second order equation with data on the parabolic line*, Canad. J. Math. **6** (1954), 542–553. MR **16**, 255.

7. L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, 1966. MR **34** #1570.

8. S. A. Tersenov, *A problem with data given on a line of degeneracy for a system of hyperbolic equations*, Dokl. Akad. Nauk SSSR **155** (1964), 285–288 = Soviet Math. Dokl. **5** (1964), 409–413. MR **29** #1439.

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