

A STABILITY THEOREM FOR A REAL ANALYTIC SINGULAR CAUCHY PROBLEM

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ABSTRACT. In this paper we prove the equation $u_{tt} - t^{2p}u_{xx} - a(t)u_x = 0$, $p > 0$, with initial conditions $u(x, 0) = \alpha(x)$, $u_t(x, 0) = \beta(x)$ is well posed provided that $\alpha(x)$ and $\beta(x)$ belong to special classes of real analytic functions. In general this problem is not stable for $p > 1$ and $\alpha(x)$ and $\beta(x)$ real analytic functions.

1. Introduction. Let $A(R)$ be the class of all real valued functions which are represented by power series expansions on the interval $(-R, R)$. In general the Cauchy problem

$$(1) \quad u_{tt} - t^{2p}u_{xx} - a(t)u_x = 0, \quad p > 0,$$

$$(2) \quad u(x, 0) = \alpha(x), \quad u_t(x, 0) = \beta(x),$$

is not well posed if $\alpha(x)$ and $\beta(x)$ belong to $A(R)$. In fact for $p > 1$ the stability in the uniform metric may be violated (for an example see [1]).

In this paper we define special classes $H(R)$ of functions in $A(R)$ and prove a theorem giving a well-posed problem provided $\alpha(x)$ and $\beta(x)$ are restricted to belong to such a class $H(R)$. M. H. Protter [6] gave a condition for a more general problem which implies that (1), (2) is well posed if $\lim_{t \rightarrow 0^+} t^{1-p}a(t) = 0$. For further papers on problems of this nature see ([1], [3], [4], [5], [8]). For general abstract existence and uniqueness theorems see [2] and [3]. In particular A. B. Nersesjan in [5] states a theorem which shows that the Cauchy problem is well posed in the case that z is a complex variable, $\alpha(z)$ and $\beta(z)$ are analytic for $|z| < R$ and solutions are admitted in the class of functions $u(z, t)$ such that, for each fixed t , $u(z, t)$ is analytic for $|z| < R$. We shall see that stability occurs for $p > 1$ in the complex variable case, as opposed to the real variable case, because of the availability of the Cauchy estimates. Namely if z is complex and

$$\max_{0 \leq |z| \leq \rho} \left| \sum_{n=0}^{\infty} \alpha_n z^n - \sum_{n=0}^{\infty} \phi_n z^n \right| = \delta$$

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then for each n , $|\alpha_n - \beta_n| \leq \delta/\rho^n$. Of course such estimates are not available in the case that x is a real variable; however, we define a more general estimate of this nature which we call a G estimate and show that in the real analytic case we have stability provided that $\alpha(x)$ and $\beta(x)$ are restricted to a class of functions which have the G estimate property.

2. Definitions and preliminaries. Let I be an index set and let $H(R) = \{\alpha_h(x)\}_{h \in I}$ be a class of functions in $A(R)$.

DEFINITION. $H(R)$ will be said to have the G estimate property if the following condition is satisfied: Suppose $0 < \rho < R$ and $\alpha(x) = \sum_{n=0}^{\infty} \alpha_n x^n$ and $\phi(x) = \sum_{n=0}^{\infty} \phi_n x^n$ are any two functions in $H(R)$. Let

$$\delta(\rho) = \max_{0 \leq |x| \leq \rho} |\alpha(x) - \phi(x)|;$$

then for every nonnegative integer n ,

$$|\alpha_n - \phi_n| \leq g(n, \rho)\delta(\rho)/\rho^n$$

where $g(n, \rho) \leq c(\rho) \prod_{i=0}^N (n+i)$, $c(\rho)$ positive and N a fixed positive integer.

EXAMPLES. (i) Suppose $\{b_n(h)\}$ is a sequence of nonnegative monotone increasing functions each defined for $h \geq 0$ and satisfying $b_n(h) \leq 1$. Define

$$H(1) = \left\{ \sum_{n=0}^{\infty} (-1)^n b_n(h) x^n \right\}_{h \in [0, \infty)},$$

then $H(1)$ has the G estimate property with $g(n, \rho) = 1$. It is easy to show that if $0 \leq h < k$ and z is a complex variable then

$$\left| \sum_{n=0}^{\infty} (-1)^n (b_n(k) - b_n(h)) z^n \right|$$

assumes its maximum on $|z| = \rho$ at $z = -\rho$. Then the Cauchy estimates for the disc $|z| \leq \rho$ in the complex plane give the required results for the interval $|x| \leq \rho$ on the real line.

(ii) A simple example in which $g(n, \rho)$ must be a function of n is

$$H(1) = \{h/(1 + x^2)^2\}_{h \in [0, \infty)}.$$

(iii) If $H(1) = \{\sin hx\}_{h \in [0, \infty)}$ we have a class of functions which can never satisfy the G estimate property.

DEFINITION. Suppose $0 < r < \rho < R$, $0 < \theta$, $0 < X$,

$$X + \theta^{p+1}/(p + 1) = r,$$

and $S = \{(x, t) : 0 \leq t \leq \theta, |x| \leq X\}$. The Cauchy problem (1), (2) is said to be G stable on S with respect to the class of functions $H(R)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\alpha_1(x), \alpha_2(x), \beta_1(x), \beta_2(x)$, belong to $H(R)$ and $\max_{0 \leq |x| \leq \rho} |\alpha_1(x) - \alpha_2(x)| < \delta, \max_{0 \leq |x| \leq \rho} |\beta_1(x) - \beta_2(x)| < \delta$, then if $u_i(x, t)$ is the solution of (1), (2) with $\alpha(x) = \alpha_i(x), \beta(x) = \beta_i(x), i = 1, 2$, it follows that

$$\max_{(x,t) \in S} |u_1(x, t) - u_2(x, t)| < \varepsilon.$$

LEMMA.

$$\sum_{n=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} g(n+w+v) \frac{(n+w+v)!}{n! w! v!} \alpha^n \beta^v \frac{\gamma^w}{w!}$$

converges for all γ if $|\alpha| + |\beta| < 1$.

PROOF. Choose $\delta > 0$ such that $|\alpha| + |\beta| + \delta < 1$. Then there exists w_0 such that for all $w \geq w_0, |\gamma|^w/w! < \delta^w$. Now for each fixed $w < w_0$ the double series

$$\sum_{n=0}^{\infty} \sum_{v=0}^{\infty} g(n+w+v) \frac{(n+w+v)!}{n! v!} \alpha^n \beta^v$$

converges. To prove this we observe that the Appell series

$$\sum_{n=0}^{\infty} \sum_{v=0}^{\infty} \frac{(n+w+v)!}{n! v!} (\alpha^n \beta^v)$$

converges for $|\alpha| + |\beta| < 1$ and the proof will be unaltered by the term $g(n+w+v)$ since $g(n) \leq c(\rho) \prod_{i=0}^N (n+i)$ (for the proof of the convergence of the Appell series see [7, pp. 210-213]). For $w \geq w_0$ the triple series is dominated by

$$\sum_{n=0}^{\infty} \sum_{w=w_0}^{\infty} \sum_{v=0}^{\infty} g(n+w+v) \frac{(n+w+v)!}{n! w! v!} |\alpha|^n |\beta|^n \delta^w$$

and again, as above, convergence follows since $|\alpha| + |\beta| + \delta < 1$ (see Lauricella functions [7, p. 227]).

3. THEOREM. If the class of functions $H(R)$ has the G estimate property then the Cauchy problem (1), (2) is G stable on S with respect to $H(R)$.

PROOF. Suppose $\alpha_1(x) - \alpha_2(x) = \sum_{n=0}^{\infty} \alpha_n x^n,$

$$\beta_1(x) - \beta_2(x) = \sum_{n=0}^{\infty} \beta_n x^n, \quad 0 < r < \rho < R,$$

and $X + \theta^{p+1}/(p+1) = r$. We need only assume that $a(t)$ is continuous.

Integrating (1) twice with respect to t , we obtain

$$(3) \quad u - Ku_{xx} - Lu_x = \alpha(x) + t\beta(x)$$

where the operators K and L are defined by

$$(Kf)(t) = \int_0^t \int_0^r s^{2p} f(s) ds dr, \quad (Lf)(t) = \int_0^t \int_0^r a(s) f(s) ds dr,$$

for $f(t)$ a real valued continuous function defined on an interval $0 \leq t \leq \theta$.

We shall denote by $I(w, v)$ the summation over all distinct operators obtained by applying L w times and K v times and define $f_n(t) = \alpha_n + \beta_n t$. Let $\| \cdot \|$ denote the supremum norm on $C[0, \theta]$. Then by the G estimate property

$$\|f_n\| \leq |\alpha_n| + |\beta_n| \theta \leq (1 + \theta)g(n)\delta/\rho^n$$

where for convenience we use the abbreviations $g(n) = g(n, \rho)$ and $\delta = \delta(\rho)$. Substitution in (3) shows there will be a solution of the form $u(x, t) = \sum_{n=0}^{\infty} a_n(t)x^n$ if, for each n ,

$$(4) \quad a_n(t) = (n+1)(n+2)Ka_{n+2}(t) + (n+1)La_{n+1}(t) + f_n(t).$$

Assuming for the moment that the infinite sum is absolutely convergent we will show that (4) is satisfied if, for all n ,

$$a_n(t) = \sum_{w=0}^{\infty} \sum_{v=0}^{\infty} \frac{(n+w+2v)!}{n!} I(w, v) f_{n+w+2v}(t).$$

It can be seen that

$$\text{for } w \geq 1, v \geq 1, I(w, v) = KI(w, v-1) + LI(w-1, v)$$

$$\text{for } w=0, v \geq 1, I(0, v) = KI(0, v-1)$$

$$\text{for } w \geq 1, v=0, I(w, 0) = LI(w-1, 0).$$

Then it may be verified directly that (4) is an identity.

It remains to show the infinite sum converges absolutely and to prove G stability on $S = \{(x, t), |x| \leq X, 0 \leq t \leq \theta\}$.

We see that $I(w, v)$ consists of $\binom{w+v}{v}$ operators each obtained by applying L w times and K v times. Further, each operator has norm at most $\theta^{2w+2v(p+1)} \|a\|^w / (2w)! E(v)$ where

$$E(v) = \prod_{j=1}^v (2pj + 2j - 1)(2pj + 2j) \geq (p+1)^{2v} (2v)!.$$

Hence, if we set $T = \sum_{n=0}^{\infty} \|a_n\| X^n$ and denote $\sum_{n=0}^{\infty} \sum_{w=0}^{\infty} \sum_{v=0}^{\infty}$ by \sum ,

$$\begin{aligned} T &\leq \sum \binom{w+v}{v} \frac{(n+w+2v)! X^n \theta^{2w+2v(p+1)} \|a\|^w (1+\theta) g(n+w+2v) \delta}{n! (2w)! (2v)! (p+1)^{2v} \rho^{n+w+2v}} \\ &\leq \sum \binom{w+v}{v} \frac{(n+w+v)!}{n! w! v!} \left(\frac{X}{\rho}\right)^n \left(\frac{\theta^{(p+1)}}{(p+1)\rho}\right)^v \frac{1}{w!} \left(\frac{\theta^2 \|a\|}{\rho}\right)^w \\ &\quad \times (1+\theta) g(n+w+v) \delta. \end{aligned}$$

Now there exists γ such that $0 < \gamma < 1$ and

$$(5) \quad X/\rho + (\theta^{p+1}/(p+1)\rho)^{1-\gamma} < 1$$

and there exists $k > 0$ such that

$$(6) \quad (\theta^{p+1}/(p+1)\rho)^\gamma + k < 1.$$

By the lemma and (5)

$$\sum \frac{(n+w+v)!}{n! w! v!} g(n+w+v) \left(\frac{X}{\rho}\right)^n [(\theta^{p+1}/(p+1)\rho)^{1-\gamma}]^v \frac{(\theta^2 \|a\|/\rho k)^w}{w!}$$

converges. Hence there exists M such that for all w and v

$$\begin{aligned} M k^w &\geq \sum_{n=0}^{\infty} \frac{(n+w+v)!}{n! w! v!} g(n+w+v) \\ &\quad \times \left(\frac{X}{\rho}\right)^n [(\theta^{p+1}/(p+1)\rho)^{1-\gamma}]^v \frac{(\theta^2 \|a\|/\rho)^w}{w!}. \end{aligned}$$

Hence

$$T \leq M(1+\theta)\delta \sum_{w=0}^{\infty} \sum_{v=0}^{\infty} \binom{w+v}{v} [(\theta^{p+1}/(p+1)\rho)^\gamma]^v k^w.$$

The latter series converges by (6) and the proof is complete.

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