A CHARACTERIZATION OF \(l-1\) MATRICES

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Abstract. Another proof is given of a known characterization of infinite matrices that preserve absolutely summable sequences where the entries of the matrices are continuous linear functions from a Fréchet space into a Fréchet space. In addition, another characterization is obtained using the adjoint matrix.

1. Introduction. Throughout this paper \(E\) and \(F\) will be Fréchet spaces, i.e. complete metrizable locally convex spaces, whose topologies are generated by the sequences of seminorms \(\{p_1\}\) and \(\{q_1\}\) respectively. A sequence, \(\{x_i\}\), in \(E\) is called absolutely summable if for every \(p_j\), \(\sum_i p_j(x_i)\) is convergent. The vector space of all such sequences will be denoted by \(l^1[E]\). It follows that \(l^1[E]\) is a Fréchet space with topology generated by \(\{P_i\}\) where

\[
P_i(x) = \sum_{n=1}^{\infty} p_i(x_n).
\]

In fact, \(l^1[E]\) is then an \(FK\)-subspace of the space of all sequences in \(E\) as defined in [1], i.e. the coordinate functions are continuous from \(l^1[E]\) into \(E\).

Let the infinite matrix \(A = (A_{nk})\) have entries which are continuous linear maps from \(E\) into \(F\). If \(x = \{x_k\}\) is any sequence in \(E\) such that \(\sum_k A_{nk}(x_k)\) is convergent for each \(n\), then we say that \(y = Ax\) where \(y_n = \sum_k A_{nk}(x_k)\) for \(n = 1, 2, \ldots\). In a recent paper [5], B. Wood has characterized matrices that carry elements of \(l^1[E]\) into \(l^1[F]\) and termed such matrices \(l-1\). In this article we offer a different proof of Wood’s characterization and obtain another by introducing the adjoint concept.

2. Results. Let the infinite matrix \(A\) be as in the introduction. We will say that (*) holds if and only if

\[
\text{for each bounded set } M_x \text{ in } E \text{ and for each fixed } j \text{ there exists } K_{x,j} \text{ such that } \sum_{n=1}^{m} q_j(A_{nr}(x)) \leq K_{x,j} \text{ for all } m, \text{ all } r \text{ and all } x \text{ in } M_x.
\]

Received by the editors February 22, 1973 and, in revised form, May 21, 1973.

AMS (MOS) subject classifications (1970). Primary 40J05; Secondary 46N05.
Before continuing we pause to mention that the FK space of all sequences in $F$ will be denoted here by $s(F)$. (In [1] we called it $F(s)$.) Moreover, any matrix map between FK spaces is continuous. In addition, for any positive integer $n$, we define $I_n: E \to l^1[E]$ by $I_n(x)$ is that sequence with $x$ in the $n$th position and zero elsewhere. It is clear that $I_n$ is continuous and linear.

**Theorem 2.1.** Let $A: l^1[E] \to s(F)$. Then $A$ is l-l if and only if (*) holds.

**Proof.** Let $A$ be l-l and let $M_\alpha$ be a bounded subset of $E$. Thus for each positive integer $i$ there exists a positive $\alpha_i$ such that $p_i(x) \leq \alpha_i$ for all $x$ in $M_\alpha$. Let

$$U_\alpha = \{ x \in l^1[E] : p_i(x) \leq \alpha_i, \ i \in \mathbb{N} \}$$

where $\mathbb{N}$ denotes the positive integers. Then $U_\alpha$ is a bounded subset of $l^1[E]$, and for any $n$ and any $x \in M_\alpha$, $I_n(x) \in U_\alpha$. Since $A$ is continuous into $l^1[F]$, $A[U_\alpha]$ is a bounded subset of $l^1[F]$. So for any fixed $j$, there exists $K_{x,j}$ such that $Q_j(y) \leq K_{x,j}$ for all $y$ in $A[U_\alpha]$. In particular, $Q_j(A(I_n(x))) \leq K_{x,j}$ for all $p$ and all $x$ in $M_\alpha$. We thus have

$$\sum_{r=1}^{\infty} q_j(A_{r,p}(x)) \leq K_{x,j} \text{ for all } p, \text{ all } x \in M_\alpha.$$

Then

$$\sum_{r=1}^{m} q_j(A_{r,p}(x)) \leq K_{x,j} \text{ for all } p, \text{ all } x \in M_\alpha \text{ and all } m,$$

i.e. (*) holds.

Now suppose (*) holds. Recall that $l^1 \otimes E$ is densely embedded in $l^1[E]$ by $\xi \otimes x \to \{ \xi \otimes x \}_{i=1}^{\infty}$, see p. 183 of [3]. In the following we will identify $\xi \otimes x$ with this sequence.

Fix $\xi \otimes x$ in $l^1[E]$. By (*), for fixed $p$ in $\mathbb{N}$ there exists $K_{x,p}$ such that $\sum_{n=1}^{m} q_p(A_{n,j}(x)) < K_{x,p}$ for all $m$ and all $j$. Fix $m$ and $r$ positive integers. Then

$$\sum_{j=1}^{r} \sum_{n=1}^{m} |\xi_j| q_p(A_{n,j}(x)) \leq \sum_{j=1}^{r} |\xi_j| \left( \sum_{n=1}^{m} q_p(A_{n,j}(x)) \right) \leq K_{x,p} \| \xi \|_{l_1}.$$

Hence

$$\sum_{j=1}^{r} \sum_{n=1}^{\infty} |\xi_j| q_p(A_{n,j}(x)) \leq K_{x,p} \| \xi \|_{l_1},$$

and also

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |\xi_j| q_p(A_{n,j}(x)) \leq K_{x,p} \| \xi \|_{l_1}.$$
It follows that

\[ \sum_{n=1}^{\infty} q_p \left( \sum_{j=1}^{\infty} \xi_j A_{n,j}(x) \right) \leq K_{x,p} \|x\|_{l^1}, \]

i.e. \( A(\xi \otimes x) \) is in \( l^1[F] \) and so \( A \) carries every element of \( l^1 \otimes E \) into \( l^1[F] \).

Let \( T \) denote \( A \) restricted to \( l^1 \otimes E \) as embedded in \( l^1[E] \). We will now show that \( T \) is continuous on \( l^1 \otimes E \) with the topology induced by \( l^1[E] \). Consider that \( T \) induces the unique bilinear mapping \( \hat{T}: l^1 \times E \rightarrow l^1[F] \) where \( \hat{T}(\xi, x) = T(\xi \otimes x) \), see p. 404 of [4]. We then have the commutative diagram

\[
\begin{array}{ccc}
l^1 \times E & \xrightarrow{T} & l^1[F] \\
\downarrow \Phi & & \downarrow T \\
l^1 \otimes E & \end{array}
\]

Notice that the topology induced on \( l^1 \otimes E \) by \( l^1[E] \) is the projective (or \( \Pi \)) topology, see p. 183 of [3]. Moreover, by Proposition 43.4, p. 438 of [4], \( T \) is continuous if and only if \( \hat{T} \) is continuous and \( \hat{T} \) is continuous if and only if it is continuous at \((0, 0)\).

Let \((\xi^n, x_n) \rightarrow 0 \) in \( l^1 \times E \). Thus \( \xi^n \rightarrow 0 \) in \( l^1 \) and \( x_n \rightarrow 0 \) in \( E \). Fix \( p \) in \( N \).

By (*) there exists \( K_p \), dependent upon the bounded set \( \{x_n: n \in N\} \) such that \( \sum_{n=1}^{\infty} q_p(A_r(x_n)) \leq K_p \) for all \( r \) and all \( n \). Thus for any \( n \),

\[
Q_p(\hat{T}(\xi^n, x_n)) = \sum_r q_p((\hat{T}(\xi^n, x_n))_r) = \sum_r q_p((T(\xi^n \otimes x_n))_r)
= \sum_r q_p \left( \sum_j A_r(\xi^n_j x_n) \right) \leq \sum_r \sum_j |\xi^n_j| q_p(A_r(x_n))
= \sum_j |\xi^n_j| \left( \sum_r q_p(A_r(x_n)) \right) \leq K_p \|\xi^n\|_{l^1}.
\]

The latter tends to zero as \( n \) tends to infinity. Thus \( \hat{T} \) is continuous at \((0, 0)\), and so \( T \) is continuous on \( l^1 \otimes E \) under the relative topology induced by \( l^1[E] \).

By continuity we may extend \( T \) to \( T': l^1[E] \rightarrow l^1[F] \). Since \( T' \) is continuous into \( l^1[F] \), it is also continuous into \( s(F) \). Now \( A \) is also continuous into \( s(F) \) and \( A \) agrees with \( T' \) on a dense subset of \( l^1[E] \). Hence, \( A \) agrees with \( T' \) on \( l^1[E] \) and therefore \( A \) carries \( l^1[E] \) into \( l^1[F] \).

We now turn to a study of the adjoint matrix. If \( A \) is an infinite matrix, \( A = (A_{ij}) \), with each \( A_{ij} \) a continuous linear map from \( E \) into \( F \), then let \( A^* = (A_{ji}^*) \) where \( A_{ji}^* \) is the adjoint of \( A_{ij} \).
Recall that the dual of $l^1[E]$ can be identified with a space of sequences, namely the set of all equicontinuous sequences of continuous linear functionals on $E$, see p. 180 of [3]. In the following we shall use this identification.

**Lemma 2.2.** Let $A$ be $l$-l. Then $A^*$ represents the adjoint of the operator defined by $A$, call it $T_A$, in the sense that given any $f = \{f_n\}$ in the dual of $l^1[F]$ and given any $x = \{x_n\}$ in $l^1[E]$, we have

$$\langle x, T^*_A f \rangle = \sum_j \left( \sum_n A^*_n f_n \langle x_j, f \rangle \right) = \sum_j \langle x_j, (A^*f)_j \rangle$$

where $\sum_n A^*_n f_n$ is convergent in the $w^*$-topology on $E'$, the dual of $E$.

**Proof.** We will first show that $\sum_n A^*_n f_n$ is pointwise convergent and consequently, by the Banach-Steinhaus closure theorem, is a continuous linear functional on $E$. Fix $j$ and fix $x$ in $E$. Since $I_j x$ is in $l^1[E]$, we have

$$\langle A(I_j x), f \rangle = \sum_n \langle A_n(x), f_n \rangle = \sum_n \langle x, A^*_n f_n \rangle.$$

For any fixed $m$, define $U_m: l^1[E] \to l^1[E]$ by $U_m(x) = \{x_1, x_2, \cdots, x_m, 0, 0, 0, \cdots\}$. It is easy to see that $U_m(x) \to x$ as $m \to \infty$ for each $x$ in $l^1[E]$. Fix $x = \{x_n\}$ in $l^1[E]$ and consider

$$\langle x, T^*_A f \rangle = \lim_m \langle U_m x, T^*_A f \rangle = \lim_m \langle A(U_m x), f \rangle$$

$$= \lim_m \sum_{n=1}^{\infty} \left( \sum_{j=1}^{m} A^*_n x_j, f_n \right)$$

$$= \lim \lim_{p \to \infty} \sum_{n=1}^{p} \sum_{j=1}^{m} \langle A^*_n x_j, f_n \rangle$$

$$= \lim \lim \sum_{n=1}^{m} \sum_{j=1}^{p} \langle A^*_n x_j, f_n \rangle$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \langle x_j, A^*_n f_n \rangle = \sum_j \left( \sum_n A^*_n f_n \langle x_j, f \rangle \right)$$

As a corollary to the previous lemma one obtains the fact that

$$\sum_{j} \sum_n \langle x_j, A^*_n f_n \rangle = \sum_j \sum_n \langle x_j, A^*_n f_n \rangle$$

by computing $\langle Ax, f \rangle$. 
Before continuing we pause to note that if $X$ is a locally convex, sequentially complete linear topological space then a subset of the dual space is strongly bounded if and only if it is $w^*$-bounded, see 18.5 of [2].

**Theorem 2.3.** Let $A : l^1[E] \to s(F)$. Then $A$ is $l$-$l$ if and only if for every $f = \{f_k\}$ in $(l^1[F])'$, the collection $\{\sum_{p=1}^m A_{p}^*f_p : m$ and $j$ positive integers\} is $w^*$-bounded, i.e. bounded in $w(E', E)$.

**Proof.** Let $A$ be $l$-$l$ and fix $f = \{f_k\}$ in $(l^1[F])'$. By the lemma, $A^*$ is the adjoint of $A$ and so $A^*$ is $w^*$-continuous. Since $U_m f \to f$ in the $w^*$-topology of $(l^1[F])'$, $A^*(U_m f) \to A^*f$ in the $w^*$-topology of $(l^1[E])'$. In particular, $(A^*(U_m f))_{m=1}^\infty$ is $w^*$-bounded in $(l^1[E])'$. Since $l^1[E]$ is sequentially complete, it is also strongly bounded. Fix $x$ in $E$. Then $\{I_j(x) : j$ a positive integer\} is a bounded subset of $\ell^\infty[F]$. Thus there is a positive $M$ such that $|A^*(U_m f)(I_j(x))| \leq M$ for all $m$ and all $j$. It follows that $|\sum_{p=1}^m A_{p}^*f_p(x)| \leq M$ for all $m$ and all $j$ so the collection mentioned in the theorem is $w^*$-bounded.

Now let the condition on $A^*$ hold. To see that $A$ is $l$-$l$ we will apply (*). Fix $q_r$ over $F$ and $M_\alpha$ a bounded subset of $E$. Consider that

$$\sum_{n=1}^m q_r(A_n(x)) = \sum_{n=1}^m q_r((A(I_j(x))) = q_r(U_m \circ A \circ I_j(x)).$$

Thus we need to show that the set

$$B = \{U_m \circ A \circ I_j(x) : j$ and $m$ in $N, x$ in $M_\alpha\}$$

is bounded in $l^1[F]$. Fix $f$ in the dual of $l^1[F]$. Then

$$f(U_m \circ A \circ I_j(x)) = \sum_{p=1}^m f_p(A_{p}^*f_p(x)) = \sum_{p=1}^m A_{p}^*f_p(x).$$

By our hypothesis there is a positive $M$ such that $|\sum_{p=1}^m A_{p}^*f_p(x)| \leq M$ for all $m$, all $j$ and all $x$ in $M_\alpha$. Thus $B$ is a weakly bounded subset of $l^1[F]$ and therefore a bounded subset of $l^1[F]$. It follows that (*) holds and $A$ is $l$-$l$.

Recall that in the scalar situation, i.e. $E$ and $F$ are the scalars and the entries of $A$ are scalars, $A$ is $l$-$l$ if and only if the transpose of $A$ is $m$-$m$, where $m$ denotes the space of all bounded scalar sequences. An analogue does exist in our case. The following result is easily established.

**Lemma 2.4.** Let $A : m(E) \to s(F)$. Then $A$ is $m$-$m$ if and only if for each bounded set $M_\alpha$ in $E$ and each fixed $j$ there exists $K_{\alpha,j}$ such that $q_j(\sum_{k=1}^\infty A_{nk}x_k) \leq K_{\alpha,j}$ for all $n$ and all $x_k$ in $M_\alpha$. 

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COROLLARY 2.5. Let $X$ and $Y$ be Banach spaces and let $A: l^1[X] \rightarrow s(Y)$. Then $A$ is $l$-$l$ if and only if $A^*$ is $m$-$m$, i.e. $A^*: m(Y') \rightarrow m(X')$.

PROOF. Let $A$ be $l$-$l$ and let $M_x$ be a bounded subset of $Y'$, say $\|f\| \leq K$ for all $f$ in $M_x$. Fix $x$ in $X$ with $\|x\| \leq 1$. Since $A$ is $l$-$l$ there exists $P$ such that $\sum_{n=1}^{\infty} \|A_{kn}(x)\| \leq P$ for all $n$. Then

$$\left| \sum_{k=1}^{\infty} A^*_k f_k(x) \right| = \left| \sum_{k=1}^{\infty} f_k A_{kn}(x) \right| \leq K \sum_{k=1}^{\infty} \|A_{kn}(x)\| \leq KP$$

for all $f_k$ in $M_x$ and all $n$. Thus for each $n$, $\sum_{k=1}^{\infty} A^*_k f_k$ is pointwise convergent and so is in $X'$. Moreover, $\|\sum_{k=1}^{\infty} A^*_k f_k\| \leq KP$ for all $f_k$ in $M_x$ and all $n$. Hence $A^*$ is $m$-$m$.

Now let $A^*$ be $m$-$m$ and let $f = \{f_k\} \in m(Y') = (l^1[Y])'$. Then $M_f = \{0\} \cup \{f_k: k$ a positive integer$\}$ is a bounded subset of $Y'$. By the lemma there is a $K_x$ such that $\|\sum_{k=1}^{n} A^*_k f_k\| \leq K_x$ for all $n$ and all $m$. It follows from 2.3 that $A$ is $l$-$l$.

REFERENCES


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