A CHARACTERIZATION OF \( l^1 \) MATRICES

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Abstract. Another proof is given of a known characterization of infinite matrices that preserve absolutely summable sequences where the entries of the matrices are continuous linear functions from a Fréchet space into a Fréchet space. In addition, another characterization is obtained using the adjoint matrix.

1. Introduction. Throughout this paper \( E \) and \( F \) will be Fréchet spaces, i.e. complete metrizable locally convex spaces, whose topologies are generated by the sequences of seminorms \( \{ p_i \} \) and \( \{ q_i \} \) respectively. A sequence, \( \{ x_i \} \), in \( E \) is called absolutely summable if for every \( p_j \), \( \sum_i p_j(x_i) \) is convergent. The vector space of all such sequences will be denoted by \( l^1[E] \). It follows that \( l^1[E] \) is a Fréchet space with topology generated by \( \{ P_i \} \) where

\[
P_i(x) = \sum_{n=1}^{\infty} p_i(x_n).
\]

In fact, \( l^1[E] \) is then an \( FK \)-subspace of the space of all sequences in \( E \) as defined in [1], i.e. the coordinate functions are continuous from \( l^1[E] \) into \( E \).

Let the infinite matrix \( A=(A_{nk}) \) have entries which are continuous linear maps from \( E \) into \( F \). If \( x=\{x_k\} \) is any sequence in \( E \) such that \( \sum_k A_{nk}(x_k) \) is convergent for each \( n \), then we say that \( y=Ax \) where \( y_n=\sum_k A_{nk}(x_k) \) for \( n=1, 2, \ldots \). In a recent paper [5], B. Wood has characterized matrices that carry elements of \( l^1[E] \) into \( l^1[F] \) and termed such matrices \( l^1 \). In this article we offer a different proof of Wood's characterization and obtain another by introducing the adjoint concept.

2. Results. Let the infinite matrix \( A \) be as in the introduction. We will say that (*) holds if and only if

\[
\text{(*) for each bounded set } M_a \text{ in } E \text{ and for each fixed } j \text{ there exists } K_{x,j},
\]

such that \( \sum_{n=1}^{m} q_j(A_{nr}(x)) \leq K_{x,j} \) for all \( m \), all \( r \) and all \( x \) in \( M_a \).

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Before continuing we pause to mention that the FK space of all sequences in $F$ will be denoted here by $s(F)$. (In [1] we called it $F(s)$.) Moreover, any matrix map between FK spaces is continuous. In addition, for any positive integer $n$, we define $I_n : E \rightarrow l^n[F]$ by $I_n(x)$ is that sequence with $x$ in the $n$th position and zero elsewhere. It is clear that $I_n$ is continuous and linear.

**Theorem 2.1.** Let $A : l^1[E] \rightarrow s(F)$. Then $A$ is $l$-$l$ if and only if (*) holds.

**Proof.** Let $A$ be $l$-$l$ and let $M_x$ be a bounded subset of $E$. Thus for each positive integer $i$ there exists a positive $\alpha_i$ such that $p_i(x) \leq \alpha_i$ for all $x$ in $M_x$. Let

$$U_x = \{x \in l^1[E] : p_i(x) \leq \alpha_i, i \in N\}$$

where $N$ denotes the positive integers. Then $U_x$ is a bounded subset of $l^1[E]$, and for any $n$ and any $x \in M_x$, $I_n(x) \in U_x$. Since $A$ is continuous into $l^1[F]$, $A[U_x]$ is a bounded subset of $l^1[F]$. So for any fixed $j$, there exists $K_{x,j}$ such that $Q_j(y) \leq K_{x,j}$ for all $y$ in $A[U_x]$. In particular, $Q_j(A(I_p(x))) \leq K_{x,j}$ for all $p$ and all $x$ in $M_x$. We thus have

$$\sum_{r=1}^\infty q_r(A_{rp}(x)) \leq K_{x,j} \quad \text{for all } p, \text{ all } x \in M_x.$$. Then

$$\sum_{r=1}^m q_r(A_{rp}(x)) \leq K_{x,j} \quad \text{for all } p, \text{ all } x \in M_x \text{ and all } m,$$

i.e. (*) holds.

Now suppose (*) holds. Recall that $l^1 \otimes E$ is densely embedded in $l^1[E]$ by $\xi \otimes x \mapsto \{\xi_i x_i\}^\infty_{i=1}$, see p. 183 of [3]. In the following we will identify $\xi \otimes x$ with this sequence.

Fix $\xi \otimes x$ in $l^1[E]$. By (*), for fixed $p$ in $N$ there exists $K_{x,j}$ such that $\sum_{n=1}^m q_n(A_{np}(x)) < K_{x,j}$ for all $m$ and all $j$. Fix $m$ and $r$ positive integers. Then

$$\sum_{j=1}^r \sum_{n=1}^m |\xi_j| q_n(A_{nj}(x)) \leq \sum_{j=1}^r |\xi_j| \left(\sum_{n=1}^m q_n(A_{nj}(x))\right) \leq K_{x,p} \|\xi\|_{l^1}.$$ 

Hence

$$\sum_{j=1}^r \sum_{n=1}^\infty |\xi_j| q_n(A_{nj}(x)) \leq K_{x,p} \|\xi\|_{l^1},$$

and also

$$\sum_{j=1}^\infty \sum_{n=1}^\infty |\xi_j| q_n(A_{nj}(x)) \leq K_{x,p} \|\xi\|_{l^1}.$$
It follows that
\[ \sum_{n=1}^{\infty} q_p \left( \sum_{j=1}^{\infty} \xi_j A_{n,j}(x) \right) \leq K_{x,p} \|x\|_1, \]
i.e. \( A(\xi \otimes x) \) is in \( l^1[F] \) and so \( A \) carries every element of \( l^1 \otimes E \) into \( l^1[F] \).

Let \( T \) denote \( A \) restricted to \( l^1 \otimes E \) as embedded in \( l^1[E] \). We will now show that \( T \) is continuous on \( l^1 \otimes E \) with the topology induced by \( l^1[E] \). Consider that \( T \) induces the unique bilinear mapping \( \hat{T}: l^1 \times E \to l^1[F] \) where \( \hat{T}((\xi, x)) = T(\xi \otimes x) \), see p. 404 of [4]. We then have the commutative diagram

\[
\begin{array}{ccc}
l^1 \times E & \xrightarrow{T} & l^1[F] \\
\Phi \downarrow & & \downarrow \Phi \\
l^1 \otimes \Phi E & \xrightarrow{T} & \Phi[F]
\end{array}
\]

Notice that the topology induced on \( l^1 \otimes E \) by \( l^1[E] \) is the projective (or \( \Pi \)) topology, see p. 183 of [3]. Moreover, by Proposition 43.4, p. 438 of [4], \( T \) is continuous if and only if \( \hat{T} \) is continuous and \( \hat{T} \) is continuous if and only if it is continuous at \((0, 0)\).

Let \((\xi^n, x_n) \to 0 \) in \( l^1 \times E \). Thus \( \xi^n \to 0 \) in \( l^1 \) and \( x_n \to 0 \) in \( E \). Fix \( p \in N \). By (*) there exists \( K_p \), dependent upon the bounded set \( \{x_n : n \in N\} \) such that \( \sum_{r=1}^{\infty} q_p(A_{r,n}(x_n)) \leq K_p \) for all \( r \) and all \( n \). Thus for any \( n \),

\[
Q_p(\hat{T}(\xi^n, x_n)) = \sum_r q_p((\hat{T}(\xi^n, x_n))_r) = \sum_r q_p((T(\xi^n \otimes x_n))_r) = \sum_r q_p \left( \sum_j A_{r,j}(\xi^n_j x_n) \right) \leq \sum_r \sum_j |\xi^n_j| q_p(A_{r,j}(x_n)) = \sum_j |\xi^n_j| \left( \sum_r q_p(A_{r,j}(x_n)) \right) \leq K_p \|\xi^n\|_1.
\]

The latter tends to zero as \( n \) tends to infinity. Thus \( \hat{T} \) is continuous at \((0, 0)\), and so \( T \) is continuous on \( l^1 \otimes E \) under the relative topology induced by \( l^1[E] \).

By continuity we may extend \( T \) to \( T': l^1[E] \to l^1[F] \). Since \( T' \) is continuous into \( l^1[F] \), it is also continuous into \( s(F) \). Now \( A \) is also continuous into \( s(F) \) and \( A \) agrees with \( T' \) on a dense subset of \( l^1[E] \). Hence, \( A \) agrees with \( T' \) on \( l^1[E] \) and therefore \( A \) carries \( l^1[E] \) into \( l^1[F] \).

We now turn to a study of the adjoint matrix. If \( A \) is an infinite matrix, \( A = (A_{ij}) \), with each \( A_{ij} \) a continuous linear map from \( E \) into \( F \), then let \( A^* = (A^*_{ji}) \) where \( A^*_{ji} \) is the adjoint of \( A_{ij} \).
Recall that the dual of $l^1[E]$ can be identified with a space of sequences, namely the set of all equicontinuous sequences of continuous linear functionals on $E$, see p. 180 of [3]. In the following we shall use this identification.

**Lemma 2.2.** Let $A$ be $l$-$l$. Then $A^*$ represents the adjoint of the operator defined by $A$, call it $T_A$, in the sense that given any $f=\{f_n\}$ in the dual of $l^1[F]$ and given any $x=\{x_n\}$ in $l^1[E]$, we have

$$\langle x, T_A^*f \rangle = \sum_j \left( \sum_n A^*_{nj}f_n \right) = \sum_j \langle x_j, (A^*f)_j \rangle$$

where $\sum_n A^*_{nj}f_n$ is convergent in the $w^*$-topology on $E'$, the dual of $E$.

**Proof.** We will first show that $\sum_n A^*_{nj}f_n$ is pointwise convergent and consequently, by the Banach-Steinhaus closure theorem, is a continuous linear functional on $E$. Fix $j$ and fix $x$ in $E$. Since $I_jx$ is in $l^1[E]$, we have

$$\langle A(I_jx), f \rangle = \sum_n \langle A_n(x), f_n \rangle = \sum_n \langle x_n, A_n^*f_n \rangle.$$

For any fixed $m$, define $U_m : l^1[E] \to l^1[E]$ by $U_m(x) = \{x_1, x_2, \ldots, x_m, 0, 0, 0, \ldots\}$. It is easy to see that $U_m(x) \to x$ as $m \to \infty$ for each $x$ in $l^1[E]$. Fix $x=\{x_n\}$ in $l^1[E]$ and consider

$$\langle x, T_A^*f \rangle = \lim_m \langle U_m x, T_A^*f \rangle = \lim_m \langle A(U_m x), f \rangle$$

$$= \lim_m \sum_n \left( \sum_{j=1}^m A_{nj}x_j, f_n \right)$$

$$= \lim_m \sum_{n=1}^m \sum_{j=1}^p \langle A_{nj}x_j, f_n \rangle$$

$$= \lim_m \sum_{n=1}^m \sum_{j=1}^p \langle A_{nj}x_j, f_n \rangle$$

$$= \sum_{j=1}^\infty \sum_{n=1}^\infty \langle x_j, A^*_{nj}f_n \rangle = \sum_j \left( \sum_n A^*_{nj}f_n \right) = \sum_j \langle x_j, (A^*f)_j \rangle.$$

As a corollary to the previous lemma one obtains the fact that

$$\sum_j \sum_n \langle x_j, A^*_{nj}f_n \rangle = \sum_j \sum_n \langle x_j, A^*_{nj}f_n \rangle$$

by computing $\langle Ax, f \rangle$. 
Before continuing we pause to note that if \( X \) is a locally convex, sequentially complete linear topological space then a subset of the dual space is strongly bounded if and only if it is \( w^* \)-bounded, see 18.5 of [2].

**Theorem 2.3.** Let \( A: l^1[E] \rightarrow s(F) \). Then \( A \) is \( l-l \) if and only if for every \( f = \{ f_k \} \) in \( (l^1[F])' \), the collection \( \{ \sum_{p=1}^{m} A_{pi}^* f_p : m \) and \( j \) positive integers \( \} \) is \( w^* \)-bounded, i.e. bounded in \( w(E', E) \).

**Proof.** Let \( A \) be \( l-l \) and fix \( f = \{ f_k \} \) in \( (l^1[F])' \). By the lemma, \( A^* \) is the adjoint of \( A \) and so \( A^* \) is \( w^* \)-continuous. Since \( U_m f \rightarrow f \) in the \( w^* \)-topology of \( (l^1[F])' \), \( A^*(U_m f) \rightarrow A^* f \) in the \( w^* \)-topology of \( (l^1[E])' \). In particular, \( \{ A^*(U_m f) \}_{m=1}^{\infty} \) is \( w^* \)-bounded in \( (l^1[E])' \). Since \( l^1[E] \) is sequentially complete, it is also strongly bounded. Fix \( x \) in \( E \). Then \( \{ I_j(x) : j \) a positive integer \( \} \) is a bounded subset of \( \phi[E] \). Thus there is a positive \( M \) such that \( |(A^*(U_m f))(I_j(x))| \leq M \) for all \( m \) and all \( j \). It follows that \( \{ \sum_{p=1}^{m} A_{pi}^* f_p(x) \} \leq M \) for all \( m \) and all \( j \) so the collection mentioned in the theorem is \( w^* \)-bounded.

Now let the condition on \( A^* \) hold. To see that \( A \) is \( l-l \) we will apply (*) Fix \( q_r \) over \( F \) and \( M_a \) a bounded subset of \( E \). Consider that

\[
\sum_{n=1}^{m} q_r(A_{n1}(x)) = \sum_{n=1}^{m} q_r((A(I_j x))_n) = q_r(U_m \circ A \circ I_j(x)).
\]

Thus we need to show that the set

\[
B = \{ U_m \circ A \circ I_j(x) : j \) and \( m \) in \( N, x \) in \( M_a \}\)

is bounded in \( l^1[F] \). Fix \( f \) in the dual of \( l^1[F] \). Then

\[
f(U_m \circ A \circ I_j(x)) = \sum_{p=1}^{m} f_p(A_{pj}(x)) = \sum_{p=1}^{m} A_{pj}^* f_p(x).
\]

By our hypothesis there is a positive \( M \) such that \( |\sum_{p=1}^{m} A_{pj}^* f_p(x)| \leq M \) for all \( m \), all \( j \) and all \( x \) in \( M_a \). Thus \( B \) is a weakly bounded subset of \( l^1[F] \) and therefore a bounded subset of \( l^1[F] \). It follows that (*) holds and \( A \) is \( l-l \).

Recall that in the scalar situation, i.e. \( E \) and \( F \) are the scalars and the entries of \( A \) are scalars, \( A \) is \( l-l \) if and only if the transpose of \( A \) is \( m-m \), where \( m \) denotes the space of all bounded scalar sequences. An analogue does exist in our case. The following result is easily established.

**Lemma 2.4.** Let \( A: m(E) \rightarrow s(F) \). Then \( A \) is \( m-m \) if and only if for each bounded set \( M_a \) in \( E \) and each fixed \( j \) there exists \( K_{a,j} \) such that \( q_j(\sum_{k=1}^{\infty} A_{nk} x_k) \leq K_{a,j} \) for all \( n \) and all \( x_k \) in \( M_a \).
Corollary 2.5. Let $X$ and $Y$ be Banach spaces and let $A : l^1[X] \to s(Y)$. Then $A$ is $l$-$l$ if and only if $A^*$ is $m$-$m$, i.e. $A^* : m(Y') \to m(X')$.

Proof. Let $A$ be $l$-$l$ and let $M_a$ be a bounded subset of $Y'$, say $\|f\| \leq K$ for all $f$ in $M_a$. Fix $x$ in $X$ with $\|x\| \leq 1$. Since $A$ is $l$-$l$ there exists $P$ such that $\sum_{k=1}^{\infty} \|A_{kn}(x)\| \leq P$ for all $n$. Then

$$\left| \sum_{k=1}^{\infty} A^*_{kn} f_k(x) \right| = \left| \sum_{k=1}^{\infty} f_k A_{kn}(x) \right| \leq K \sum_{k=1}^{\infty} \|A_{kn}(x)\| \leq KP$$

for all $f_k$ in $M_a$ and all $n$. Thus for each $n$, $\sum_{k=1}^{\infty} A^*_{kn} f_k$ is pointwise convergent and so is in $X'$. Moreover, $\|\sum_{k=1}^{\infty} A^*_{kn} f_k\| \leq KP$ for all $f_k$ in $M_a$ and all $n$. Hence $A^*$ is $m$-$m$.

Now let $A^*$ be $m$-$m$ and let $f = \{f_k\} \in m(Y') = (l^1[Y])'$. Then $M_a = \{0\} \cup \{f_k : k$ a positive integer$\}$ is a bounded subset of $Y'$. By the lemma there is a $K_a$ such that $\|\sum_{k=1}^{m} A^*_{kn} f_k\| \leq K_a$ for all $n$ and all $m$. It follows from 2.3 that $A$ is $l$-$l$.

References


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