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A CHARACTERIZATION OF l-l MATRICES

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ABSTRACT. Another proof is given of a known characterization of infinite matrices that preserve absolutely summable sequences where the entries of the matrices are continuous linear functions from a Fréchet space into a Fréchet space. In addition, another characterization is obtained using the adjoint matrix.

1. **Introduction.** Throughout this paper E and F will be Fréchet spaces, i.e. complete metrizable locally convex spaces, whose topologies are generated by the sequences of seminorms $\{p_i\}$ and $\{q_i\}$ respectively. A sequence, $\{x_i\}$, in E is called absolutely summable if for every p_j, $\sum_i p_j(x_i)$ is convergent. The vector space of all such sequences will be denoted by $l^1[E]$. It follows that $l^1[E]$ is a Fréchet space with topology generated by $\{P_i\}$ where

$$P_i(x) = \sum_{n=1}^{\infty} p_i(x_n).$$

In fact, $l^1[E]$ is then an FK-subspace of the space of all sequences in E as defined in [1], i.e. the coordinate functions are continuous from $l^1[E]$ into E.

Let the infinite matrix $A=(A_{nk})$ have entries which are continuous linear maps from E into F. If $x=\{x_k\}$ is any sequence in E such that $\sum_k A_{nk}(x_k)$ is convergent for each n, then we say that $y=Ax$ where $y_n=\sum_k A_{nk}(x_k)$ for $n=1, 2, \cdots$. In a recent paper [5], B. Wood has characterized matrices that carry elements of $l^1[E]$ into $l^1[F]$ and termed such matrices l-l. In this article we offer a different proof of Wood's characterization and obtain another by introducing the adjoint concept.

2. **Results.** Let the infinite matrix A be as in the introduction. We will say that (*) holds if and only if

(*) for each bounded set M_α in E and for each fixed j there exists $K_{\alpha,j}$ such that $\sum_{n=1}^{m} q_j(A_{nr}(x)) \leqq K_{\alpha,j}$ for all m, all r and all x in M_α.

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Before continuing we pause to mention that the FK space of all sequences in F will be denoted here by $s(F)$. (In [1] we called it $F(s)$.) Moreover, any matrix map between FK spaces is continuous. In addition, for any positive integer n, we define $I_n: E \to l^1[E]$ by $I_n(x)$ is that sequence with x in the nth position and zero elsewhere. It is clear that I_n is continuous and linear.

THEOREM 2.1. *Let $A: l^1[E] \to s(F)$. Then A is 1-1 if and only if* (*) *holds.*

PROOF. Let A be 1-1 and let M_α be a bounded subset of E. Thus for each positive integer i there exists a positive α_i such that $p_i(x) \leqq \alpha_i$ for all x in M_α. Let

$$U_\alpha = \{x \in l^1[E] : P_i(x) \leqq \alpha_i,\ i \in N\}$$

where N denotes the positive integers. Then U_α is a bounded subset of $l^1[E]$, and for any n and any $x \in M_\alpha$, $I_n(x) \in U_\alpha$. Since A is continuous into $l^1[F]$, $A[U_\alpha]$ is a bounded subset of $l^1[F]$. So for any fixed j, there exists $K_{\alpha,j}$ such that $Q_j(y) \leqq K_{\alpha,j}$ for all y in $A[U_\alpha]$. In particular, $Q_j(A(I_p(x))) \leqq K_{\alpha,j}$ for all p and all x in M_α. We thus have

$$\sum_{r=1}^{\infty} q_j(A_{rp}(x)) \leqq K_{\alpha,j} \quad \text{for all } p, \text{ all } x \in M_\alpha.$$

Then

$$\sum_{r=1}^{m} q_j(A_{rp}(x)) \leqq K_{\alpha,j} \quad \text{for all } p, \text{ all } x \in M_\alpha \text{ and all } m,$$

i.e. (*) holds.

Now suppose (*) holds. Recall that $l^1 \otimes E$ is densely embedded in $l^1[E]$ by $\xi \otimes X \to \{\xi_i x\}_{i=1}^{\infty}$, see p. 183 of [3]. In the following we will identify $\xi \otimes x$ with this sequence.

Fix $\xi \otimes x$ in $l^1[E]$. By (*), for fixed p in N there exists $K_{x,p}$ such that $\sum_{n=1}^{m} q_p(A_{nj}(x)) < K_{x,p}$ for all m and all j. Fix m and r positive integers. Then

$$\sum_{j=1}^{r} \sum_{n=1}^{m} |\xi_j|\, q_p(A_{nj}(x)) \leqq \sum_{j=1}^{r} |\xi_j| \left(\sum_{n=1}^{m} q_p(A_{nj}(x)) \right) \leqq K_{x,p}\, \|\xi\|_{l_1}.$$

Hence

$$\sum_{j=1}^{r} \sum_{n=1}^{\infty} |\xi_j|\, q_p(A_{nj}(x)) \leqq K_{x,p}\, \|\xi\|_{l_1},$$

and also

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |\xi_j|\, q_p(A_{nj}(x)) \leqq K_{x,p}\, \|\xi\|_{l_1}.$$

It follows that

$$\sum_{n=1}^{\infty} q_p\left(\sum_{j=1}^{\infty} \xi_j A_{nj}(x)\right) \leqq K_{x,p} \|\xi\|_{l_1},$$

i.e. $A(\xi\otimes x)$ is in $l^1[F]$ and so A carries every element of $l^1\otimes E$ into $l^1[F]$.

Let T denote A restricted to $l^1\otimes E$ as embedded in $l^1[E]$. We will now show that T is continuous on $l^1\otimes E$ with the topology induced by $l^1[E]$. Consider that T induces the unique bilinear mapping $\hat{T}: l^1\times E \to l^1[F]$ where $\hat{T}((\xi, x)) = T(\xi\otimes x)$, see p. 404 of [**4**]. We then have the commutative diagram

$$\begin{tikzcd} l^1 \times E \arrow[r, "\hat{T}"] \arrow[d, "\Phi"'] & l^1[F] \\ l^1 \otimes_{\Pi} E \arrow[ur, "T"'] & \end{tikzcd}$$

Notice that the topology induced on $l^1\otimes E$ by $l^1[E]$ is the projective (or Π) topology, see p. 183 of [**3**]. Moreover, by Proposition 43.4, p. 438 of [**4**], T is continuous if and only if $\hat{T}$ is continuous and $\hat{T}$ is continuous if and only if it is continuous at $(0, 0)$.

Let $(\xi^n, x_n)\to 0$ in $l^1\times E$. Thus $\xi^n\to 0$ in l^1 and $x_n\to 0$ in E. Fix p in N. By (*) there exists K_p, dependent upon the bounded set $\{x_n : n \text{ in } N\}$ such that $\sum_{j=1}^{\infty} q_p(A_{jr}(x_n))\leqq K_p$ for all r and all n. Thus for any n,

$$\begin{aligned} Q_p(\hat{T}(\xi^n, x_n)) &= \sum_r q_p((\hat{T}(\xi^n, x_n))_r) = \sum_r q_p((T(\xi^n \otimes x_n))_r) \\ &= \sum_r q_p\left(\sum_j A_{rj}(\xi^n_j x_n)\right) \leqq \sum_r \sum_j |\xi^n_j|\, q_p(A_{rj}(x_n)) \\ &= \sum_j |\xi^n_j| \left(\sum_r q_p(A_{rj}(x_n))\right) \leqq K_p \|\xi^n\|_{l_1}. \end{aligned}$$

The latter tends to zero as n tends to infinity. Thus $\hat{T}$ is continuous at $(0, 0)$, and so T is continuous on $l^1\otimes E$ under the relative topology induced by $l^1[E]$.

By continuity we may extend T to $T' : l^1[E]\to l^1[F]$. Since T' is continuous into $l^1[F]$, it is also continuous into $s(F)$. Now A is also continuous into $s(F)$ and A agrees with T' on a dense subset of $l^1[E]$. Hence, A agrees with T' on $l^1[E]$ and therefore A carries $l^1[E]$ into $l^1[F]$.

We now turn to a study of the adjoint matrix. If A is an infinite matrix, $A = (A_{ij})$, with each A_{ij} a continuous linear map from E into F, then let $A^* = (A^*_{ji})$ where A^*_{ji} is the adjoint of A_{ji}.

Recall that the dual of $l^1[E]$ can be identified with a space of sequences, namely the set of all equicontinuous sequences of continuous linear functionals on E, see p. 180 of [3]. In the following we shall use this identification.

LEMMA 2.2. *Let A be l-l. Then A^* represents the adjoint of the operator defined by A, call it T_A, in the sense that given any $f=\{f_n\}$ in the dual of $l^1[F]$ and given any $x=\{x_n\}$ in $l^1[E]$, we have*

$$\langle x, T_A^* f\rangle = \sum_j \left\langle x_j, \sum_n A_{nj}^* f_n \right\rangle = \sum_j \langle x_j, (A^*f)_j\rangle$$

where $\sum_n A_{nj}^* f_n$ is convergent in the w-topology on E', the dual of E.*

PROOF. We will first show that $\sum_n A_{nj}^* f_n$ is pointwise convergent and consequently, by the Banach-Steinhaus closure theorem, is a continuous linear functional on E. Fix j and fix x in E. Since $I_j x$ is in $l^1[E]$, we have

$$\langle A(I_j x), f\rangle = \sum_n \langle A_{nj}(x), f_n\rangle = \sum_n \langle x, A_{nj}^* f_n\rangle.$$

For any fixed m, define $U_m: l^1[E]\to l^1[E]$ by $U_m(x)=\{x_1, x_2, \cdots, x_m, 0, 0, 0, \cdots\}$. It is easy to see that $U_m(x)\to x$ as $m\to\infty$ for each x in $l^1[E]$. Fix $x=\{x_n\}$ in $l^1[E]$ and consider

$$\begin{aligned}
\langle x, T_A^* f\rangle &= \lim_m \langle U_m x, T_A^* f\rangle = \lim_m \langle A(U_m x), f\rangle \\
&= \lim_m \sum_{n=1}^{\infty} \left\langle \sum_{j=1}^{m} A_{nj} x_j, f_n \right\rangle \\
&= \lim_m \lim_p \sum_{n=1}^{p} \sum_{j=1}^{m} \langle A_{nj} x_j, f_n\rangle \\
&= \lim_m \lim_p \sum_{j=1}^{m} \sum_{n=1}^{p} \langle A_{nj} x_j, f_n\rangle \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \langle x_j, A_{nj}^* f_n\rangle = \sum_j \left\langle x_j, \sum_n A_{nj}^* f_n \right\rangle \\
&= \sum_j \langle x_j, (A^* f)_j\rangle.
\end{aligned}$$

As a corollary to the previous lemma one obtains the fact that

$$\sum_j \sum_n \langle x_j, A_{nj}^* f_n\rangle = \sum_n \sum_j \langle x_j, A_{nj}^* f_n\rangle$$

by computing $\langle Ax, f\rangle$.

Before continuing we pause to note that if X is a locally convex, sequentially complete linear topological space then a subset of the dual space is strongly bounded if and only if it is w^*-bounded, see 18.5 of [2].

THEOREM 2.3. *Let $A:l^1[E]\to s(F)$. Then A is l-l if and only if for every $f=\{f_k\}$ in $(l^1[F])'$, the collection $\{\sum_{p=1}^m A_{pj}^* f_p : m$ and j positive integers$\}$ is w^*-bounded, i.e. bounded in $w(E', E)$.*

PROOF. Let A be l-l and fix $f=\{f_k\}$ in $(l^1[F])'$. By the lemma, A^* is the adjoint of A and so A^* is w^*-continuous. Since $U_m f\to f$ in the w^*-topology of $(l^1[F])'$, $A^*(U_m f)\to A^* f$ in the w^*-topology of $(l^1[E])'$. In particular, $\{A^*(U_m f)\}_{m=1}^\infty$ is w^*-bounded in $(l^1[E])'$. Since $l^1[E]$ is sequentially complete, it is also strongly bounded. Fix x in E. Then $\{I_j(x): j$ a positive integer$\}$ is a bounded subset of $l^1[E]$. Thus there is a positive M such that $|(A^*(U_m f))(I_j(x))|\leqq M$ for all m and all j. It follows that $|\sum_{p=1}^m A_{pj}^* f_p(x)|\leqq M$ for all m and all j so the collection mentioned in the theorem is w^*-bounded.

Now let the condition on A^* hold. To see that A is l-l we will apply (*). Fix q_r over F and M_α a bounded subset of E. Consider that

$$\sum_{n=1}^m q_r(A_{nj}(x)) = \sum_{n=1}^m q_r((A(I_j x))_n) = Q_r(U_m \circ A \circ I_j(x)).$$

Thus we need to show that the set

$$B = \{U_m \circ A \circ I_j(x) : j \text{ and } m \text{ in } N,\ x \text{ in } M_\alpha\}$$

is bounded in $l^1[F]$. Fix f in the dual of $l^1[F]$. Then

$$f(U_m \circ A \circ I_j(x)) = \sum_{p=1}^m f_p(A_{pj}(x)) = \sum_{p=1}^m A_{pj}^* f_p(x).$$

By our hypothesis there is a positive M such that $|\sum_{p=1}^m A_{pj}^* f_p(x)|\leqq M$ for all m, all j and all x in M_α. Thus B is a weakly bounded subset of $l^1[F]$ and therefore a bounded subset of $l^1[F]$. It follows that (*) holds and A is l-l.

Recall that in the scalar situation, i.e. E and F are the scalars and the entries of A are scalars, A is l-l if and only if the transpose of A is m-m, where m denotes the space of all bounded scalar sequences. An analogue does exist in our case. The following result is easily established.

LEMMA 2.4. *Let $A:m(E)\to s(F)$. Then A is m-m if and only if for each bounded set M_α in E and each fixed j there exists $K_{\alpha,j}$ such that $q_j(\sum_{k=1}^\infty A_{nk}x_k)\leqq K_{\alpha,j}$ for all n and all x_k in M_α.*

COROLLARY 2.5. *Let X and Y be Banach spaces and let $A: l^1[X] \to s(Y)$. Then A is l-l if and only if A^* is m-m, i.e. $A^*: m(Y') \to m(X')$.*

PROOF. Let A be l-l and let M_α be a bounded subset of Y', say $\|f\| \leqq K$ for all f in M_α. Fix x in X with $\|x\| \leqq 1$. Since A is l-l there exists P such that $\sum_{k=1}^\infty \|A_{kn}(x)\| \leqq P$ for all n. Then

$$\left|\sum_{k=1}^{\infty} A^*_{kn} f_k(x)\right| = \left|\sum_{k=1}^{\infty} f_k A_{kn}(x)\right| \leqq K \sum_{k=1}^{\infty} \|A_{kn}(x)\| \leqq KP$$

for all f_k in M_α and all n. Thus for each n, $\sum_{k=1}^\infty A^*_{kn} f_k$ is pointwise convergent and so is in X'. Moreover, $\|\sum_{k=1}^\infty A^*_{kn} f_k\| \leqq KP$ for all f_k in M_α and all n. Hence A^* is m-m.

Now let A^* be m-m and let $f = \{f_k\} \in m(Y') = (l^1[Y])'$. Then $M_\alpha = \{0\} \cup \{f_k : k \text{ a positive integer}\}$ is a bounded subset of Y'. By the lemma there is a K_α such that $\|\sum_{k=1}^m A^*_{kn} f_k\| \leqq K_\alpha$ for all n and all m. It follows from 2.3 that A is l-l.

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