

COMMUTATIVE QF-1 RINGS

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ABSTRACT. If R is a commutative artinian ring, then it is known that every faithful R -module is balanced (i.e. has the double centralizer property) if and only if R is a quasi-Frobenius ring. In this note it is shown that the assumption on R to be artinian can be replaced by the weaker condition that R is noetherian.

Let R be an associative ring with 1. The R -module M is called balanced if the canonical ring homomorphisms from R into the double centralizer of M is surjective. It is well known that every faithful module over a quasi-Frobenius ring is balanced. Recently, several authors proved the converse for commutative artinian rings [5], [2], [1]. The aim of this note is to show that the assumption on R to be artinian can be weakened.

THEOREM. *Let R be a commutative noetherian ring. Then every faithful R -module is balanced if and only if R is a quasi-Frobenius ring.*

We recall that a commutative ring R is quasi-Frobenius if R is the direct product of finitely many local artinian rings with simple socles. The essence of the Theorem is that a commutative ring with all faithful modules balanced is artinian iff it is noetherian. It is an open problem whether the same is true in the noncommutative case. The proof of the Theorem uses several, by now well-known, constructions of nonbalanced modules (see, e.g. [4]), as well as the following lemma which goes back to N. Divinsky [3].

LEMMA 1. *A commutative, noetherian, subdirect irreducible ring R is a local quasi-Frobenius ring.*

PROOF. Let S be the socle of R , then the annihilator M of S is a maximal ideal. If x is a zero-divisor of R , say $xy=0$ for some $y \neq 0$, then $xS \subseteq xyR=0$, thus $x \in M$. This implies that R can be embedded in the quotient ring $\hat{R}=R_M$ of R with the elements of $R \setminus M$ as denominators. The proper ideals of \hat{R} are in one-to-one correspondence to the ideals of R contained in M ; in particular, \hat{R} is noetherian and subdirect irreducible. If \hat{M} is the unique maximal ideal of \hat{R} , then by Krull's intersection theorem,

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the intersection of all powers \hat{M}^n is zero. Let \hat{S} be the socle of \hat{R} . Since $\hat{S} \neq 0$, there is some natural number n with $\hat{S} \not\subseteq \hat{M}^n$, and this implies $\hat{M}^n = 0$. We conclude from $M \subseteq \hat{M}$, that M is nilpotent. As a consequence, R is a local ring with a nilpotent radical. It is well known that such a ring is noetherian if and only if it is artinian. Since, by assumption, R has a simple socle, R is a quasi-Frobenius ring.

Following R. M. Thrall [6], we will call the ring R a QF-1 ring, if every finitely generated faithful module is balanced. Let us call R a strong QF-1 ring if every faithful module (not only the finitely generated ones) is balanced. It should be noted that some authors (e.g. [1]) use the notion of a QF-1 ring in the latter sense.

V. P. Camillo [1] has shown that a commutative, strong QF-1 ring R is the product of a finite number of local, strong QF-1 rings, provided there are no nonzero maps between the injective hulls of distinct simple R -modules. As a consequence of Lemma 1, this condition is always satisfied for noetherian rings.

LEMMA 2. *Let R be a commutative noetherian ring. Then there are no nonzero maps between the injective hulls of two nonisomorphic simple R -modules.*

PROOF. Let S and T be nonisomorphic simple R -modules, and ES and ET their injective hulls, respectively. Assume there is a nonzero homomorphism $f: ES \rightarrow ET$. Let $x \in ES$ be an element such that xf is a nonzero element of the socle T of ET . Then Rx has an epimorphic image isomorphic to T . Let A be the annihilator of x , thus $R/A \approx Rx$. Since ES has a minimal submodule which is contained in every nontrivial submodule, we know that R/A is a subdirect irreducible ring. By Lemma 1, R/A is a local ring. But this implies that all simple R/A -modules are isomorphic. Both S and T may be considered as R/A -modules, since S is a submodule and T a factor module of R/A . This contradicts the fact that S and T were assumed to be nonisomorphic R -modules.

Thus, we may restrict our attention to local, strong QF-1 rings. If R is a local ring, we denote by W its radical.

LEMMA 3. *A local, commutative, noetherian, strong QF-1 ring R has a nontrivial socle.*

PROOF. Since R is noetherian, every ideal is finitely generated; thus $W = Rx_1 + \cdots + Rx_k$ for certain elements x_1, \cdots, x_k . Assume R has no socle, then the R -homomorphism $\varphi: R \rightarrow \bigoplus_{i=1}^k R$ defined by $r \rightarrow (rx_1, \cdots, rx_k)$ is a monomorphism, and the image is contained in $\bigoplus_{i=1}^k W$, the radical of $\bigoplus_{i=1}^k R$. Define M_n and $\varphi_n: M_n \rightarrow M_{n+1}$ inductively by $M_1 = R$, $M_2 = \bigoplus_{i=1}^k R$, $\varphi_1 = \varphi$ and $M_{n+1} = \bigoplus_{i=1}^k M_n$, $\varphi_n = \bigoplus_{i=1}^k \varphi_{n-1}$. Then all

φ_n 's are monomorphisms, and the image of φ_n is contained in the radical of M_{n+1} . Let M be the direct limit of the diagram

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} \dots,$$

and, since all φ_n 's are monomorphisms, we may assume that the M_n 's are submodules of M and that the φ_n 's are inclusion maps. It follows from $M = \bigcup M_n$ and the fact that all M_n 's have zero socle, that M also has zero socle. Assume that X is a maximal submodule of M . Obviously, $M_n \not\subseteq X$ for some n . If $m \in M_n \setminus X$, then we have

$$(X \cap M_{n+1}) + Rm = (X + Rm) \cap M_{n+1} = M \cap M_{n+1} = M_{n+1}.$$

But $m = m\varphi_n$ belongs to the radical of M_{n+1} . This implies that $X \cap M_{n+1} = M_{n+1}$, thus $M_n \subseteq M_{n+1} \subseteq X$, a contradiction. We have shown that the faithful module M has neither a minimal nor a maximal submodule. According to [1], this contradicts the assumption that R is a strong QF-1 ring.

LEMMA 4. *A local, commutative QF-1 ring R with nonzero socle is a subdirect irreducible ring.*

PROOF. Let S be a minimal ideal of R . We show that every nonzero ideal I contains S . Assume I is a proper nonzero ideal, and $S \cap I = 0$. Take elements $0 \neq s \in S$ and $0 \neq x \in I$, and consider the module $M = (R \oplus R)/R(s, x)$. The endomorphisms of M may be lifted to endomorphisms φ of $R \oplus R$ mapping $R(s, x)$ into itself, thus to matrices

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

with $\alpha_{ij} \in R$ and

$$(*) \quad (s, x) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = (s\alpha_{11} + x\alpha_{21}, s\alpha_{12} + x\alpha_{22}) = (rs, rx)$$

for some $r \in R$. If

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

satisfies (*), then $x\alpha_{21} = rs - s\alpha_{11} \in Rx \cap Rs = 0$, thus α_{21} belongs to the radical W of R . Similarly, $s\alpha_{12} = rx - x\alpha_{22} \in Rx \cap Rs = 0$ implies that $\alpha_{12} \in W$. Let us define an additive homomorphism ψ' of $R \oplus R$ into itself by $(r_1, r_2) \rightarrow (0, sr_2)$. Since s is in the socle of R , and $R(s, x) \subseteq W \oplus W$, ψ' maps $R(s, x)$ into 0, and therefore induces an additive endomorphism

ψ of M . Now ψ' commutes with all matrices

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

satisfying (*), since

$$[\psi'(r_1, r_2)] \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = (0, sr_2) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = (sr_2\alpha_{21}, sr_2\alpha_{22})$$

and

$$\begin{aligned} \psi' \left[(r_1, r_2) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \right] &= \psi'(r_1\alpha_{11} + r_2\alpha_{21}, r_1\alpha_{12} + r_2\alpha_{22}) \\ &= (0, sr_1\alpha_{12} + sr_2\alpha_{22}) \end{aligned}$$

are equal in view of the fact that α_{12} and α_{21} belong to W , whereas s is an element of the socle of R . This shows that ψ belongs to the double centralizer of M . On the one hand, $\psi \neq 0$, since $\psi'(0, 1) = (0, s) \notin R(s, x)$; on the other hand, ψ vanished on the submodule

$$[R(1, 0) + R(s, x)]/R(s, x) \approx R(1, 0)/[R(1, 0) \cap R(s, x)] = R(1, 0).$$

Thus, ψ is not induced by multiplication.

We have shown that a local, commutative, noetherian, strong QF-1 ring R is subdirect irreducible. Using again Lemma 1, we see that such a ring R is in fact a quasi-Frobenius ring. This proves the theorem.

ADDED IN PROOF (November 1973). An independent (and quite different) proof of the theorem has appeared in the meanwhile by H. Storrer in his paper *Epimorphic extensions of non-commutative rings*, Comment. Math. Helv. **48** (1973), 72–86.

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