PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

A GENERALIZATION OF STRONG RIESZIAN SUMMABILITY

L. I. HOLDER

ABSTRACT. Strong summability, $[\alpha, \beta; p]$, for the Bosanquet-Linfoot (α, β) summability method is defined so that $[\alpha, 0; p]$ is identical to strong Rieszian summability, $[R; \alpha, p]$. The main result proved in this paper shows consistency in the sense that $[\alpha, \beta; p]$ summability implies $[\alpha', \beta'; q]$ summability, for $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$; and $1 \leqq q \leqq p$. Also, a necessary condition for $[\alpha, \beta; p]$ summability and relationships between strong and absolute (α, β) summability are given.

1. **Introduction.** In 1931 L. S. Bosanquet and E. H. Linfoot [1] gave the following definition of a double scale summability method, generalizing Riesz' arithmetic mean (R, n, α).

DEFINITION 1.1. A series $\sum a_n$ is said to be summable (α, β) where either $\alpha > 0$ or $\alpha = 0$, $\beta \geqq 0$, to the sum S, provided

$$\lim_{\omega \to \infty} \sum_{n < \omega} B(1 - n/\omega)^{\alpha} \log^{-\beta} \frac{C}{1 - n/\omega} a_n = S \tag{1.1}$$

for each sufficiently large C, where $B = \log^{\beta} C$.

Absolute summability for this method has been defined [2] and has been shown to be consistent, in the sense that $|\alpha, \beta|$ summability implies $|\alpha', \beta'|$ summability if either $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$, where $\alpha > 0$ or $\alpha = 0$, $\beta \geqq 0$. In the present paper a similar result is obtained for strong (α, β) summability.

DEFINITION 2.1. The series $\sum a_n$ is said to be strongly summable (α, β) with index $p \geqq 1$, briefly, summable $[\alpha, \beta; p]$ to the sum S, if $\sum a_n$ is summable (α, β) to S, and for all sufficiently large C,

$$\int_1^{\omega} \left| u \frac{d}{du} A^{(\alpha,\beta)}(u) \right|^p du = o(\omega) \tag{1.2}$$

Received by the editors November 20, 1972 and, in revised form, April 23, 1973.
AMS (MOS) subject classifications (1970). Primary 40F05.
Key words and phrases. Bosanquet-Linfoot summability, Rieszian summability, strong summability, consistency, absolute summability.

as $\omega \to \infty$, for $\alpha > 0$, or $\alpha = 0$, $\beta \geqq 0$, where

$$A^{(\alpha,\beta)}(u) = \sum_{n<u} B(1 - n/u)^\alpha \log^{-\beta} \frac{C}{1 - n/u} a_n. \tag{1.3}$$

For $\beta = 0$ this reduces to strong Rieszian summability as defined by Boyd and Hyslop [2].

The following additional notation will be used throughout the remainder of this paper. Let

$$\begin{aligned} \Phi_{\alpha,\beta}(u) &= Bu^\alpha \log^{-\beta}(C/u), \quad \text{if } u > 0 \\ &= 0 \qquad\qquad\qquad\quad \text{if } u = 0 \text{ and } \alpha > 0 \text{ or } \alpha = 0, \quad \beta \geqq 0, \end{aligned}$$

$$A(u) = \sum_{n \leqq u} a_n, \qquad B(u) = \sum_{n \leqq u} n a_n.$$

2. **Preliminary lemmas.**

LEMMA 2.1. *If* $\sum a_n$ *is summable* $[\alpha, \beta; p]$, *then one of the following relationships holds,*

(i) $\alpha > 1 - 1/p$;

(ii) $\alpha = 1 - 1/p$, $\beta > 1/p$ *if* $p > 1$, $\beta > 0$ *if* $p = 1$.

PROOF. For $u > 1$, write $N = [u]$ if u is not an integer, and $N = u - 1$ otherwise. Then we have

$$\begin{aligned} u \frac{d}{du} A^{(\alpha,\beta)}(u) &= u^{-1} \sum_{n=0}^{N} \Phi'_{\alpha,\beta}(1 - n/u) n a_n \\ &= u^{-1} \sum_{n=0}^{N-1} \Phi'_{\alpha,\beta}(1 - n/u) n a_n + u^{-1} \Phi'_{\alpha,\beta}(1 - N/u) N a_N \\ &= S_1 + S_2, \end{aligned}$$

say. It follows that

$$\int_1^\omega |S_2|^p \, du \leqq 2^p \left\{ \int_1^\omega \left| u \frac{d}{du} A^{(\alpha,\beta)}(u) \right|^p du + \int_1^\omega |S_1|^p \, du \right\}. \tag{2.1}$$

Now,

$$\begin{aligned} \int_1^\omega |S_2|^p \, du &= \int_1^\omega u^{-p} |\Phi'_{\alpha,\beta}(1 - N/u)|^p N^p |a_N|^p \, du \\ &\geqq \sum_{N=1}^{[\omega]-1} \int_N^{N+1} u^{-p} |\Phi'_{\alpha,\beta}(1 - N/u)|^p N^p |a_N|^p \, du. \end{aligned}$$

A series of straightforward estimates can now be employed, each of which strengthens the inequality, with the final result,

$$\int_1^\omega |S_2|^p \, du \geqq \sum_{N=1}^{[\omega]-1} C_N \int_0^{1/(N+1)} v^{\alpha p - p} \log^{-\beta p} \frac{C}{v} \, dv$$

if $\alpha>0$, and the last integral converges only if (i) or (ii) holds. A similar result is true for $\alpha=0$, $\beta>0$.

Since the second integral on the right-hand side of (2.1) is finite, the conclusion follows. This result parallels the necessary condition for strong Rieszian summability, $[R;\alpha,p]$, that $\alpha>1-1/p$, shown by Boyd and Hyslop [2].

LEMMA 2.2. *Let the integer h be defined as follows:*

$$\begin{aligned} h &= [\alpha] && \text{if } \alpha \text{ is not an integer,} \\ & && \text{or } \alpha \text{ is an integer and } \beta > 0, \\ &= \alpha - 1 && \text{if } \alpha \text{ is an integer and } \beta \leqq 0. \end{aligned}$$

Then for $\alpha>1$ or $\alpha=1$, $\beta>0$,

$$\frac{d}{du} A^{(\alpha,\beta)}(u) = u^{-h-1}\int_0^1 \Phi_{\alpha,\beta}^{(h+1)}(1-v)B_{h-1}(uv)\,dv,$$

where $B_0(x)=B(x)$ and $B_j(x)=\int_0^x B_{j-1}(t)\,dt$, $j\geqq 1$.

PROOF. We may write

$$\begin{aligned} \frac{d}{du} A^{(\alpha,\beta)}(u) &= \frac{1}{u^2}\sum_{n<u}\Phi'_{\alpha,\beta}(1-n/u)na_n = \frac{1}{u^2}\int_0^u \Phi'_{\alpha,\beta}(1-t/u)\,dB(t) \\ &= \frac{1}{u^2}\left\{[\Phi'_{\alpha,\beta}(1-t/u)B(t)]_{t=0}^{t=u} + \frac{1}{u}\int_0^u \Phi''_{\alpha,\beta}(1-t/u)B(t)\,dt\right\} \\ &= \frac{1}{u^2}\int_0^1 \Phi''_{\alpha,\beta}(1-v)B(uv)\,dv. \end{aligned}$$

This is the desired result if $h=1$ and for $h>1$ repeated integration by parts yields the result.

LEMMA 2.3. *For $\alpha>0$ or $\alpha=0$, $\beta>0$ and h defined as in Lemma 2.2 write $K(u)=\Phi_{\alpha,\beta}^{(h+1)}(1-u)$, and for $\alpha'>\alpha$ or $\alpha'=\alpha$, $\beta'>\beta$, write $k(u)=\Phi_{\alpha',\beta'}^{(h+1)}(1-u)/\Phi_{\alpha,\beta}^{(h+1)}(1-u)$. Then $K(u)$ and $k(u)$ have the following properties for sufficiently large C.*

(i) *Either $K(u)$ is a constant on $[0, 1]$, or $K(u)$ is positive, nondecreasing, continuously differentiable on $[0, 1)$, $\in L[0, 1]$, $\lim_{u\to 1^-} K(u)=+\infty$, and $uK'(u)/K(u)$ is nondecreasing on $[0, 1)$.*

(ii) *$k(u)$ is continuously differentiable on $[0, 1)$, nonincreasing, and $\lim_{u\to 1^-} k(u)=0$.*

PROOF. These results are direct consequences of the definitions and the order relations

$$\begin{aligned} K(u) &= O\{(1-u)^{\alpha-h-1}\log^{-\beta}(C/(1-u))\} && \text{if } \alpha \neq h \\ &= O\{(1-u)^{-1}\log^{-\beta-1}(C/(1-u))\} && \text{if } \alpha = h. \end{aligned}$$

3. Consistency.

THEOREM 3.1. *If $\sum a_n$ is summable $[\alpha, \beta; p]$ to S, it is also summable $[\alpha', \beta'; q]$ to S, where $1 \leqq q \leqq p$, and either $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$.*

PROOF. *Case* (i): $h \geqq 1$. From Lemmas 2.2 and 2.3 we have, for $p \geqq 1$,

$$\int_1^\omega \left| u \frac{d}{du} A^{(\alpha',\beta')}(u) \right|^p du = \int_1^\omega \left| u^{-h} \int_0^1 \Phi^{(h+1)}_{(\alpha',\beta')}(1-v) B_{h-1}(uv)\, dv \right|^p du$$
$$= \int_1^\omega \left| u^{-h} \int_0^1 K(v) k(v) B_{h-1}(uv)\, dv \right|^p du.$$

We integrate by parts, writing $g_0(u, v) = u^{-h} \int_0^v K(t) B_{h-1}(ut)\, dt$, and obtain, since the integrated term vanishes,

$$\int_1^\omega \left| u \frac{d}{du} A^{(\alpha',\beta')}(u) \right|^p du = \int_1^\omega \left| \int_0^1 (-k'(v)) g_0(u, v)\, dv \right|^p du$$
$$= \int_1^\omega \left| \int_0^1 |k'(v)|^{1-1/p} \{ |k'(v)|^{1/p} g_0(u, v) \}\, dv \right|^p du$$
$$\leqq \int_1^\omega \left\{ \left(\int_0^1 |k'(v)|\, dv \right)^{p-1} \left(\int_0^1 |k'(v)|\, |g_0(u, v)|^p \right) dv \right\} du$$
$$\leqq A \int_1^\omega \int_0^1 |k'(v)|\, |g_0(u, v)|^p\, dv\, du,$$

where $A = (k(0))^{p-1}$. The inner integral exists since, for example, when $\alpha' > \alpha$, the integrand is of order at worst $(1-v)^{\alpha'-h-1} \log^\delta (C/(1-v))$ for some δ, and $(1-v)^{-1} \log^{-(\beta'-\beta)-1}(C/(1-v))$ if $\alpha' = \alpha$, $\beta' > \beta$. So the iterated integral also exists and interchange of order of integration is justified. Thus,

$$\int_1^\omega \left| u \frac{d}{du} A^{(\alpha',\beta')}(u) \right|^p du \leqq A \int_0^1 |k'(v)|\, dv \int_1^\omega |g_0(u, v)|^p\, du$$
$$= A \int_1^\omega |g_0(u, v_0)|^p\, du \tag{3.1}$$

where $0 < v_0 < 1$, since $\int_1^\omega |g_0(u, v)|^p\, du$ is a continuous function of v on $[0, 1]$. The constant A depends only on α, β, α', β' and p and is not necessarily the same in different occurrences. Consider now the integral on the right-hand side of (3.1).

$$\int_1^\omega |g_0(u, v_0)|^p\, du = \int_1^\omega \left| u^{-h} \int_0^{v_0} K(t) B_{h-1}(ut)\, dt \right|^p du$$
$$= \int_1^\omega \left| u^{-h} v_0 \int_0^1 K(vv_0) B_{h-1}(uvv_0)\, dv \right|^p du. \tag{3.2}$$

If $K(u)$ is not a constant, we integrate by parts, writing

$$K(vv_0) = \frac{K(vv_0)}{K(v)} K(v), \qquad g_1(u, v) = u^{-h} v_0 \int_0^v K(t) B_{h-1}(utv_0)\, dt.$$

The integrated term vanishes, since $\lim_{v\to 1} K(vv_0)/K(v) = 0$, by Lemma 2.3. Thus,

$$\int_1^\omega |g_0(u, v_0)| \ du = \int_1^\omega \left| \int_0^1 \left(- \frac{d}{dv} \frac{K(vv_0)}{K(v)} \right) g_1(u, v)\, dv \right|^p du.$$

From the property that $uK'(u)/K(u)$ is nondecreasing, one can show that $d(K(vv_0)/K(v))/dv \leqq 0$ on $[0, 1)$. Applying Hölder's inequality, as above, then yields,

$$\begin{aligned} \int_1^\omega |g_0(u, v_0)|^p\, du &\leqq A \int_1^\omega \int_0^1 \left| \frac{d}{dv} \frac{K(vv_0)}{K(v)} \right| |g_1(u, v)|^p\, dv\, du \\ &= A \int_0^1 \left| \frac{d}{dv} \frac{K(vv_0)}{K(v)} \right| dv \int_1^\omega |g_1(u, v)|^p\, du \\ &= A \int_1^\omega |g_1(u, v_1)|^p\, du, \end{aligned}$$

where $0 < v_1 < 1$. Continuing in this manner, we obtain, from (3.1), for $n = 1, 2, \cdots$,

$$\int_1^\omega \left| u \frac{d}{du} A^{(\alpha', \beta')}(u) \right|^p du \leqq \int_1^\omega |g_n(u, v_n)|^p\, du, \tag{3.3}$$

where

$$g_n(u, v) = u^{-h} v_0 v_1 \cdots v_{n-1} \int_0^v K(t) B_{h-1}(utv_0 v_1 \cdots v_{n-1})\, dt,$$

$0 < v_k < 1$, $k = 0, 1, \cdots, n-1$, and the v_k depend on ω but not on u.

The sequence $\{V_n\}$, where $V_n = v_0 v_1 \cdots v_n$, is monotone decreasing and positive, hence convergent. Let $\lim V_n = V$. Also $\lim v_n = 1$. After changing variables (3.3) may be written

$$\int_1^\omega \left| u \frac{d}{du} A^{(\alpha', \beta)}(u) \right|^p du \leqq A \int_1^\omega \left| u^{-h} V_n \int_0^1 K(vv_n) B_{h-1}(uvV_n)\, dv \right|^p du,$$

and this is true for $n = 0, 1, 2, \cdots$. For each $u \in [1, \omega]$, the function $B_{h-1}(uv)$ is continuous for at least almost all $v \in [0, 1]$, being a step function if $h = 1$ and absolutely continuous if $h > 1$, and it is bounded. Since $K(v) \in L[0, 1]$ and is continuous on $[0, 1)$, it follows that passage to the limit inside the inner integral is justified. Passage to the limit inside the outer integral is also justified, since the integrand is uniformly bounded, with respect to n. Thus, if $V>0$,

$$\begin{aligned}
\int_1^\omega \left| u\frac{d}{du} A^{(\alpha',\beta')}(u)\right|^p du &\leqq A\int_1^\omega \left| u^{-h} V\int_0^1 K(v)B_{h-1}(uVv)\,dv\right|^p du \\
&= \frac{A}{V}\int_V^{V\omega} \left| t^{-h}V^{h+1}\int_0^1 K(v)B_{h-1}(tv)\,dv\right|^p dt \\
&\leqq A\int_V^\omega \left| t^{-h}\int_0^1 K(v)B_{h-1}(tv)\,dv\right|^p dt \\
&= A\int_1^\omega \left| t\frac{d}{dt} A^{(\alpha,\beta)}(t)\right|^p dt,
\end{aligned} \tag{3.4}$$

since for $0\leqq t<1$, $dA^{(\alpha,\beta)}(t)/dt=0$. So

$$\int_1^\omega \left| u\frac{d}{du} A^{(\alpha',\beta')}(u)\right|^p du = o(\omega)$$

as $\omega\to\infty$, if $V>0$ and $K(u)$ is not a constant.

If $V=0$, the result is trivial, and if $K(u)$ is a constant, then equation (3.2) gives a satisfactory estimate on $\int_1^\omega |u\, dA(u)^{(\alpha',\beta')}/du|^p\, du$. In fact, equation (3.4) will again be valid, with $V=v_0$.

Case (ii): $h=0$. Bosanquet and Linfoot **[1]** have shown that

$$A^{(\alpha,\beta)}(u) = \int_0^1 \Phi'_{\alpha,\beta}(1-v)A(uv)\,dv.$$

Let

$$\Delta_t A^{(\alpha,\beta)}(u) = A^{(\alpha,\beta)}(u+t) - A^{(\alpha,\beta)}(u)$$

and

$$\Delta_t A(u,v) = A((u+t)v) - A(uv).$$

Then, we may write

$$\int_1^\omega |u\,\Delta_t A^{(\alpha',\beta')}(u)|^p\, du = \int_1^\omega \left| u\int_0^1 K(v)k(v)\,\Delta_t A(u,v)\,dv\right|^p du.$$

Proceeding exactly as in Case (i), we arrive at the following inequality, analogous to (3.4),

$$\int_1^\omega |u\,\Delta_t A^{(\alpha',\beta')}(u)|^p\, du \leqq A\int_1^\omega \left| uV\int_0^1 K(v)\,\Delta_t A(u,vV)\,dv\right|^p du,$$

where $0\leqq V<1$. If $V=0$, the desired result is obtained after dividing by $|t|^p$ and letting t approach zero. The interchange of limit and integration is discussed below. If $V>0$, we write $\tau=Vt$, $v=uV$ and observe that

$\Delta_t A(u, vV) = \Delta_\tau A(\nu, v)$. This gives,

$$\int_1^\omega |u\,\Delta_t A^{(\alpha',\beta')}(u)|^p\,du \leqq A\int_V^{V\omega}\left|\nu\int_0^1 K(v)\,\Delta_\tau A(\nu, v)\,dv\right|^p d\nu \tag{3.5}$$
$$\leqq A\int_V^\omega |\nu\,\Delta_\tau A^{(\alpha,\beta)}(\nu)|^p\,d\nu.$$

Suppose now that $0<t<\frac{1}{2}$, so that $0<\tau<\frac{1}{2}$ also, and as $t\to 0$, $\tau\to 0$. We divide both sides of (3.5) by $|t^p|$ and consider the right-hand integral.

$$\int_V^\omega\left|\nu\frac{\Delta_\tau A^{(\alpha,\beta)}(\nu)}{t}\right|^p d\nu \leqq \int_{1/2}^\omega\left|\nu\int_0^1 K(v)\frac{\Delta_\tau A(\nu, v)}{\tau}\,dv\right|^p d\nu, \tag{3.6}$$

since $\Delta_\tau A(\nu, v)=0$ for $0\leqq\nu\leqq\frac{1}{2}$ and $0\leqq v\leqq 1$. We partition the interval $[\frac{1}{2}, \omega]$ as follows: $\{\frac{1}{2}, 1, 2, 3, \cdots, [\omega], \omega\}$. For $k<\nu<k+1$ $(k=1, 2, \cdots, [\omega]-1)$, we may write the inner integral on the right-hand side of (3.6) in the form

$$\sum_{n=1}^{k}\frac{a_n}{\tau}\int_{n/\nu+\tau}^{n/\nu} K(v)\,dv + R_{k+1},$$

where $R_{k+1}=0$ if $0<\tau\leqq k+1-\nu$, and $R_{k+1}=\tau^{-1}a_{k+1}\int_{k+1/\nu+\tau}^{1} K(v)\,dv$ if $\tau>k+1-\nu$. For $0<\tau\leqq k+1-\nu$, then, we have

$$\left|\nu\int_0^1 K(v)\frac{\Delta_\tau A(\nu, v)}{\tau}\,dv\right|^p \leqq M_k[K(k/\nu)]^p,$$

where $M_k=\max_{n\leqq k}|a_n|^p(k+1)^p$. Now,

$$\int_k^{k+1}[K(k/\nu)]^p\,d\nu \leqq \frac{(k+1)^2}{k}\int_0^1 [k(u)]^p\,du,$$

and for $0<\alpha<1$,

$$[K(u)]^p = O\{(1-u)^{\alpha p-p}\log^{-\beta p}(C/(1-u))\}.$$

Thus, by Lemma 2.1 the above integral is finite. When $\alpha=0$, the same lemma shows that $p=1$ and $\beta>0$, so that

$$[K(u)]^p = O\{(1-u)^{-1}\log^{-\beta-1}(C/(1-u))\}$$

which is again L-integrable on $[0, 1]$. The case $\alpha=1$, $\beta\leqq 0$ also results in a finite integral, as is easily seen.

When $\tau > k+1-\nu$, we need also consider the term R_{k+1}. We have

$$
\begin{aligned}
|\nu R_{k+1}|^p &\leqq A \left| \frac{\nu}{\tau} \int_{k+1/\nu+\tau}^{1} \Phi'_{\alpha,\beta}(1-v)\, dv \right|^p \\
&\leqq A \left| \frac{k+1}{\tau} \Phi_{\alpha,\beta}\left(1 - \frac{k+1}{k+1+\tau}\right) \right|^p \\
&\leqq A \left| \frac{k+1}{\tau} \Phi_{\alpha,\beta}\left(\frac{\tau}{k+1}\right) \right|^p \\
&\leqq A \left| \frac{k+1}{k+1-\nu} \Phi_{\alpha,\beta}\left(\frac{k+1-\nu}{k+1}\right) \right|^p \\
&= O\left\{ (k+1-\nu)^{\alpha p - p} \log^{-\beta p} \frac{C}{k+1-\nu} \right\},
\end{aligned}
$$

if $0<\alpha<1$, which, again by Lemma 2.1 is L-integrable on $[k, k+1]$. If $\alpha=0$, $\beta>0$ or $\alpha=1$, $\beta\leqq 0$, Lemma 2.1 and the appropriate order relations show that in these cases, too, the integral involved is finite.

The arguments for the intervals $(\frac{1}{2}, 1)$ and $([\omega], \omega)$ are similar. Thus, for $0<\tau<\frac{1}{2}$, the integrand on the right-hand side of (3.6) is uniformly dominated by a summable function. The argument for $-\frac{1}{2}<\tau<0$ is similar. So passage to the limit under the integral sign in (3.6) as $\tau\to 0$ is permitted. Since for $\alpha'>\alpha$, or $\alpha'=\alpha$, $\beta'>\beta$, convergence of the integrals involved is improved, interchange of limit and integration after dividing by $|t|^p$ on the left-hand side of (3.5) is also permitted. From (3.5), then, we have

$$
\int_1^\omega \left| u \frac{d}{du} A^{(\alpha',\beta')}(u) \right|^p du \leqq A \int_1^\omega \left| \nu \frac{d}{d\nu} A^{(\alpha,\beta)}(\nu) \right|^p d\nu.
$$

Now suppose $1\leqq q\leqq p$. Then

$$
\begin{aligned}
\int_1^\omega \left| u \frac{d}{du} A^{(\alpha',\beta')}(u) \right|^q du &\leqq \left\{ \int_1^\omega \left| u \frac{d}{du} A^{(\alpha',\beta')}(u) \right|^p \right\}^{q/p} \left\{ \int_0^\omega du \right\}^{1-q/p} \\
&= o(\omega^{q/p}) \cdot O(\omega^{1-q/p}) = o(\omega).
\end{aligned}
$$

Finally, Bosanquet and Linfoot have shown that summability (α, β) implies summability (α', β') with α, β, α', β' as specified. The proof is therefore complete.

4. **Relation to absolute summability.** Absolute (α, β) summability means that $A^{(\alpha,\beta)}(u)$ is of bounded variation on $[1, \infty)$ (see **[3]**). Examples are known of series summable $|C, \alpha|$ but not $[C, \alpha; p]$ for $p>1$ (see e.g. Hyslop **[4]**) and also of series summable $[C, \alpha; p]$ but not $|C, \alpha|$. These

same examples show, then, that $|\alpha, \beta|$ and $[\alpha, \beta; p]$ summability are not in general comparable. We do, however, have the following theorem.

THEOREM 4.1. *If* $\sum a_n$ *is summable* $|\alpha, \beta|$, *for* $\alpha>0$, *or* $\alpha=0$, $\beta>0$, *then it is also summable* $[\alpha, \beta; 1]$.

PROOF. It is known (see [3]) that $|\alpha, \beta|$ summability implies (α, β) summability. By hypothesis

$$\lim_{t\to\infty} \int_1^t \left| \frac{d}{du} A^{(\alpha,\beta)}(u) \right| du = L \qquad (0 \leqq L < \infty)$$

from which it follows also that

$$\lim_{\omega\to\infty} \frac{1}{\omega} \int_1^\omega \int_1^t \left| \frac{d}{du} A^{(\alpha,\beta)}(u) \right| du\, dt = L.$$

But

$$\frac{1}{\omega} \int_1^\omega \int_1^t \left| \frac{d}{du} A^{(\alpha,\beta)}(u) \right| du\, dt = \frac{1}{\omega} \int_1^\omega \int_u^\omega \left| \frac{d}{du} A^{(\alpha,\beta)}(u) \right| dt\, du$$
$$= \int_1^\omega \left| \frac{d}{du} A^{(\alpha,\beta)}(u) \right| du - \frac{1}{\omega} \int_1^\omega \left| u \frac{d}{du} A^{(\alpha,\beta)}(u) \right| du.$$

Letting $\omega \to \infty$, we get

$$L = L - \lim_{\omega\to\infty} \frac{1}{\omega} \int_1^\omega \left| u \frac{d}{du} A^{(\alpha,\beta)}(u) \right| du.$$

So $\sum a_n$ is summable $[\alpha, \beta; 1]$.

REFERENCES

1. L. S. Bosanquet and E. H. Linfoot, *Generalized means and the summability of Fourier series*, Quart. J. Math. Oxford Ser. **2** (1931), 207–229.

2. A. V. Boyd and J. M. Hyslop, *A definition for strong Rieszian summability and its relationship to strong Cesàro summability*, Proc. Glasgow Math. Assoc. **1** (1952), 94–99. MR **14**, 463.

3. B. J. Boyer and L. I. Holder, *A generalization of absolute Rieszian summability*, Proc. Amer. Math. Soc. **14** (1963), 459–464. MR **26** #6647.

4. J. M. Hyslop, *Note on the strong summability of series*, Proc. Glasgow Math. Assoc. **1** (1952), 16–20. MR **14**, 368.

DEPARTMENT OF MATHEMATICS, GETTYSBURG COLLEGE, GETTYSBURG, PENNSYLVANIA 17325