

## COMMUTATION PROPERTIES OF THE COEFFICIENT MATRIX IN THE DIFFERENTIAL EQUATION OF AN INNER FUNCTION

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**ABSTRACT.** Let  $A(x)$  be an operator valued function that is analytic on the real axis. Assume that  $A(x)$  is selfadjoint for each real  $x$ . It is shown that  $A(x)$  and  $\int_0^x A(s) ds$  commute for all real  $x$  iff  $A(x)$  and  $A(y)$  commute for all real  $x$  and  $y$ . This result is then used to establish several new characterizations of the Potapov inner functions of normal operators  $T$  such that  $\|T\| < 1$ . The case where  $\|T\| = 1$ ,  $r(T) < 1$  and  $A_T(x)$  and  $A_T(y)$  commute for real  $x$  and  $y$  is discussed. Here  $A_T(x) = -iU_T(x)U_T(x)^*$  and  $U_T(x)$  is the Potapov inner function for  $T$ .

This paper is a continuation of the author's study of differentiable inner functions begun in [1], [2], and [4]. It answers some of the questions raised in [1] and improves the characterization of  $A_T$  for normal  $T$  given in [4]. The results of [4] on the  $L^1$  norm of  $A_T$  have recently been improved and elaborated on by Helson [7] and Herrero [9].

**1. Main result.** The main result of this paper is the following theorem. It answers the major unresolved question left in [1]. By operator we mean a bounded linear operator acting in a fixed separable Hilbert space  $\mathcal{H}$ . If  $X$  and  $Y$  are operators, then  $[X, Y] = XY - YX$ .

**THEOREM 1.** *Suppose that  $A(x)$  is an operator valued function which is analytic on the real axis. Suppose further that  $A(x)$  is selfadjoint for every real  $x$ . Then the following are equivalent.*

- (\*)  $[A(x), A(y)] = 0$  for all real  $x$  and  $y$ .
- (\*\*)  $[A(x), \int_0^x A(y) dy] = 0$  for all real  $x$ .

To prove this result we need the following lemma.

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LEMMA 1. *If  $T_1, T_2$  are selfadjoint operators and  $[T_1, [T_1, T_2]]=0$ , then  $[T_1, T_2]=0$ .*

PROOF. Suppose  $T_1$  and  $T_2$  are selfadjoint and that  $[T_1, [T_1, T_2]]=0$ . Then  $[T_1, T_2]$  is quasinilpotent [11, p. 4]. But  $i[T_1, T_2]$  is selfadjoint since  $T_1$  and  $T_2$  are. Thus  $i[T_1, T_2]=0$  since it is a selfadjoint quasinilpotent operator.

Theorem 1 is written in the notation of [1]. For purposes of a proof, it is notationally more convenient to prove the following result which includes Theorem 1.

THEOREM 2. *Suppose that  $B(x)$  is an operator valued function that is analytic on the real axis. Suppose further that  $B(x)$  is selfadjoint for all real  $x$ . Then the following are equivalent.*

- (i)  $[B'(x), B(x)]=0$  for all real  $x$ .
- (ii)  $[B(x), B(y)]=0$  for all real  $x$  and  $y$ .

PROOF. Clearly (ii) implies (i). Assume then that (i) holds. Now  $B(x)$  admits a series expansion  $\sum_{m=0}^{\infty} B_m x^m$  near zero. Repeated differentiations with respect to  $x$  show  $B_m$  is selfadjoint for every  $m$ . Following Hellman [5] we insert the series for  $B(x)$  into  $[B'(x), B(x)]=0$  and get

- (1)  $[B_1, B_0]=0$ ,
- (2)  $2[B_2, B_0]=0$ ,
- (3)  $3[B_3, B_0]+[B_2, B_1]=0$ ,
- (4)  $4[B_4, B_0]+2[B_3, B_1]=0$ ,
- (5)  $5[B_5, B_0]+3[B_4, B_1]+[B_3, B_2]=0, \dots$

We wish to show that  $[B_i, B_j]=0$  for all  $i, j \geq 0$ . We will say (P) holds for  $B_i$  if  $[B_i, B_j]=0$  for all  $j$ . Now  $[B_n, B_0]$  is a sum of  $[B_i, B_j]$  with  $0 \leq i, j \leq n-1$ . Thus if  $[B_0, B_k]=0$  for  $k < n$ , then  $[B_0, [B_0, B_n]]=0$  and hence  $[B_0, B_n]=0$  by Lemma 1. But  $[B_1, B_0]=[B_2, B_0]=0$ . Thus (P) holds for  $B_0$  by induction. But then  $[B_2, B_1]=[B_3, B_1]=0$  by (3) and (4). Furthermore  $[B_n, B_1]$  is a linear combination of  $[B_i, B_j]$  with  $1 \leq i, j \leq n-1$ . As before we get (P) holds for  $B_1$ . Continuing in this manner we get that (P) holds for all  $B_i$ . Thus (ii) holds for all  $x$  by analytic continuation.

To see that Theorem 2 implies Theorem 1, let  $B(x)=\int_0^x A(s) ds$  where  $A(s)$  is selfadjoint. By Theorem 2,  $[\int_0^x A(s) ds, \int_0^y A(s) ds]=0$  for all  $x$  and  $y$  if and only if  $[\int_0^x A(s) ds, A(x)]=0$  for all  $x$ . But  $[\int_0^x A(s) ds, \int_0^y A(s) ds]=0$  for all  $x$  and  $y$  if and only if  $[A(x), A(y)]=0$  for all  $x$  and  $y$ . The if part of this last statement is clear. To get the only if, differentiate  $[\int_0^x A(s) ds, \int_0^y A(s) dy]=0$  first with respect to  $x$  and then with respect to  $y$ .

2. **Necessity of assumptions.** The assumption that  $A(x)$  is selfadjoint is necessary. The following example is a modification of one of Hellman's.

EXAMPLE 1. Let

$$A(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2x \\ 4x^3 & 0 & 0 & 0 \\ 9x^2 & 0 & 0 & 0 \end{pmatrix}.$$

Then a straightforward calculation verifies  $[A(x), \int_0^x A(y) dy] = 0$ . Now the coefficients of the  $x$  and  $x^2$  terms in the power series expansion of  $A$  do not commute. Hence  $[A(x), A(y)]$  is not always zero.

**3. Applications.** We review briefly the notation of [3] and [4]. Let  $T$  be an operator such that  $\|T\| \leq 1$  and the spectral radius,  $r(T)$ , is less than one. If  $U_T(w)$  is the Potapov inner function for  $T$ ,  $|w| \leq 1$ , and  $z = i(1-w)/(1+w)$ , then let  $U_T(z) = U_T(w)$ . The variable  $x$  denotes real values of  $z$ .  $U_T(z)$  satisfies the differential equation  $U_T'(x) = iA_T(x)U_T(x)$  where  $A_T(x) \geq 0$  for all  $x$ . A discussion of Potapov inner functions may be found in [10]. The differential equation was introduced in [6] and extensively studied in [3]. The connection between inner functions and operators is nicely explained in [8].

Combining the results of [1], [4], and this paper we get the following theorem.

**THEOREM 3.** *Suppose that  $\|T\| < 1$ . Let  $U_T(z)$  be the Potapov inner function of  $T$  on the upper half-plane and  $A_T(x)$  be defined by  $U_T'(x) = iA_T(x)U_T(x)$ . Then the following are equivalent.*

- (a)  $T$  is normal.
- (b)  $[U_T(x), U_T(y)] = 0$  for all real  $x$  and  $y$ .
- (c)  $[A_T(x), A_T(y)] = 0$  for all real  $x$  and  $y$ .
- (d)  $[A_T(x), \int_0^x A_T(y) dy] = 0$  for all real  $x$ .
- (e)  $U_T(x) = \exp(i \int_0^x A_T(y) dy)$ .
- (f)  $[U_T(x), U_T'(x)] = 0$  for all real  $x$ .
- (g)  $[A_T(x), U_T(x)] = 0$  for all real  $x$ .
- (h)  $[A_T(x), U_T'(x)] = 0$  for all real  $x$ .

**PROOF.** The equivalence of (a) and (b) is due to Sherman [12]. The equivalence of (d), (e), (f), (g), and (h) was shown in [1], while (a) and (c) were shown equivalent in [4]. Theorem 1 gives us that (c) and (d) are equivalent.

**4. Discussion.** It had been initially hoped in [1] that the Potapov inner function for some nonnormal operators would satisfy some type of commutation property and hence be easier to work with. Theorem 3 shows

that most of the obvious ones are equivalent to the normality of  $T$  if  $\|T\| < 1$ . If  $\|T\| = 1$  and  $r(T) < 1$ , then (c) does not necessarily imply (a), while (a) always implies (c). An example was given in [4]. That example was  $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

There remains then the possibility that Theorem 1 of [1] could be useful in studying nonspectraloid operators for which (c) holds. It would be of interest, as the next theorem shows, to characterize those operators for which (c) holds.

**THEOREM 4.** *If  $\|T\| \leq 1$ ,  $r(T) < 1$ , and  $[A_T(x), A_T(y)] = 0$  for all  $x$  and  $y$ , then  $T$  has an invariant subspace.*

**PROOF.** Recall from [1, Theorem 7], that the closure of the range of  $A_T(x)$  is independent of  $x$ . If there exists an  $x_0$  such that  $A_T(x_0)$  is not a scalar multiple of a projection, then let  $P$  be a nontrivial projection in the spectral resolution of  $A_T(x_0)$ . Then  $[A_T(x), P] = 0$  for all  $x$  and  $A_T(x) = A_T(x)P \oplus A_T(x)(I - P) = A_1(x) \oplus A_2(x)$ . Let  $U_1(x)$  and  $U_2(x)$  be the solutions of  $X' = iA_1X$ ,  $X(0) = P$ , and  $X' = iA_2X$ ,  $X(0) = (I - P)$ , respectively. Then  $U_T(x)$  and  $(U_1 \oplus (I - P))(P \oplus U_2)U_T(0)$  both satisfy the initial value problem  $X' = iA_T X$ ,  $X(0) = U_T(0)$ , and hence are equal. Thus  $U_T$  factors and  $T$  has an invariant subspace. There remains the possibility that  $A_T(x) = p(x)P$  for a scalar function  $p(x)$  and projection  $P$ . In this case  $U_T(x) = (q(x)P \oplus (I - P))U_T(0)$  where  $q$  is the scalar inner function satisfying  $q'(x) = ip(x)q(x)$ ,  $q(0) = 1$ . Clearly  $U_T$  factors unless it is trivial, that is,  $\dim P \neq 1$  and  $q$  is a single Blaschke factor.

The techniques used in proving Theorem 4 can be adapted to prove the following more general result.

**THEOREM 5.** *Suppose that  $\|T\| \leq 1$  and  $r(T) < 1$ . Let  $P$  be the projection onto the closure of the range of  $A_T(x)$ . If there exists an operator  $B$  such that  $[B, A_T(x)] = 0$  for all real  $x$  and  $BP$  is not a scalar multiple of  $P$ , then  $T$  has an invariant subspace.*

The example mentioned earlier shows that one cannot improve Theorem 5 to produce a reducing subspace since  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has none. If we assume  $\|T\| < 1$ , then Theorem 5 can be modified to prove  $T$  has a reducing subspace. See [3, p. 37] for details.

As a first step toward characterizing the nonnormal operators for which (c) holds we show they must be isometric on a nontrivial subspace.

**THEOREM 6.** *Suppose that  $1 \geq \|T\| > r(T)$  and that 1 is not an eigenvalue of  $T^*T$ . Then  $[A_T(x), A_T(y)] = 0$  for all real  $x$  and  $y$  if and only if  $T$  is normal.*

PROOF. If  $T$  is normal, the result follows from the formula for  $A_T(x)$  given in [3, p. 31] which expresses  $A_T(x)$  in terms of  $T$  and  $T^*$ . Assume then that  $1 \geq \|T\| > r(T)$  and 1 is not an eigenvalue of  $T^*T$ . The case  $\|T\| < 1$  was done in [4], so assume  $\|T\| = 1$ . Let  $D = [I - T^*T]^{1/2}$ . Then  $D$  is a one-to-one selfadjoint operator with a dense range. Up to a scalar function  $A_T(x)$  is  $D(I - wT)^{-1}(w - T^*)^{-1}D$ . Expressing this as a Laurent series convergent for  $1 > |w| > r(T)$ , we get  $\sum_{n=-\infty}^{-1} DBT^{*-n-1}Dw^n + \sum_{n=0}^{\infty} DT^{n+1}BDw^n$  where  $B = \sum_{m=0}^{\infty} T^mT^{*m}$ . By analytic continuation  $[A_T(z), A_T(u)] = 0$  for  $z, u$  such that  $1 > |(z-i)/(z+i)|, |(u-i)/(u+i)| > r(T)$ . Hence all the coefficients in the Laurent series commute. In particular,  $DT^iBD^2BD = DBD^2T^iBD$ . Hence

$$T^iBD^2 = BD^2T^i \text{ or } [T^i, BD^2] = 0.$$

Now

$$DT^iBD^2BT^{*j}D = DBT^{*j}D^2T^iBD.$$

Hence

$$T^iBD^2BT^{*j} = BT^{*j}D^2T^iB,$$

and thus

$$D^2T^iBT^{*j} = T^{*j}D^2T^iB \text{ or } [T^{*j}, D^2T^iB] = 0.$$

Setting  $i=j=1$  gives  $D^2TBT^* = T^*D^2TB$ . Multiply both sides by  $T^*$  on the right and use the identity  $TBT^* = B - I$  to get  $D^2(B - I)T^* = T^*D^2(B - I)$ . But  $D^2BT^* = T^*D^2B$  and hence  $[T^*, D^2] = 0$ . Thus  $T$  and  $T^*$  commute with  $T^*T$ , which in turn implies that  $T$  and  $T^*$  commute with  $B$  also. The earlier identity  $D^2TBT^* = T^*D^2TB$  now gives  $TT^* = T^*T$ , that is, that  $T$  is normal.

Theorem 6 improves Theorem 6 of [4] and is probably about the best possible for inner functions analytic on the closed disc.

The assumption that 1 is not an eigenvalue of  $T^*T$  is equivalent to assuming that  $\|T\phi\| < \|\phi\|$  for all nonzero  $\phi$  in  $\mathcal{H}$ .

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