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ON THE FIRST COHOMOLOGY GROUP OF DISCRETE GROUPS WITH PROPERTY (T)

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ABSTRACT. Let G be a separable locally compact group with property (T), i.e., the class of one dimensional trivial representations is an isolated point in the dual space $\hat{G}$ of G. Let $\pi: G \to O_n$ be a continuous representation of G into the orthogonal group. In this note, we show that $H^1(G, \pi)=0$.

Let G be a separable locally compact group. Let $\hat{G}$ be the set of all equivalence classes of separable irreducible unitary representations of G. We give $\hat{G}$ the inner-hull-kernel topology. $\hat{G}$ is called the dual space of G. Following [**4**], G is said to have property (T) if the equivalence class I of the one dimensional trivial representation of G is an isolated point in $\hat{G}$. Let $\pi: G \to O_n$ be a continuous representation of G into the group of all n by n orthogonal matrices. Let us write $C^1(G, \pi)$ and $B^1(G, \pi)$ for the groups of continuous 1-cocycles and 1-coboundaries respectively. Let $H^1(G, \pi)=C^1(G, \pi)/B^1(G, \pi)$. In this note, we are going to prove that $H^1(G, \pi)=0$ if G is a separable locally compact group with property (T). Applying to discrete subgroups of p-adic groups or real Lie groups, our result generalizes slightly some results in [**4**] and [**3**, Theorem 6.5]. Our argument still follows the spirit of [**4**]. Before we give the proof of the main result, we shall first establish some lemmas needed later.

LEMMA 1. *Let H be a locally compact group and $\pi: H \to O_n$ a representation. Let $\varphi: H \to R^n$ be a* 1*-cocycle. If $\varphi(H)$ is bounded in R^n, then φ is a coboundary.*

PROOF. Let K be the closure of $\pi(H)$ in O_n and $K \cdot R^n$ the semidirect product of K and R^n where R^n is the normal subgroup and K acts on R^n in the natural manner. Consider then the map $f: H \to K \cdot R^n$ defined by $h \to \pi(h)\varphi(h)$ $(h \in H)$. Since φ is a 1-cocycle, f clearly is a homomorphism. $\varphi(H)$ is bounded by assumption. It yields that $\mathrm{Cl}(f(H))$ is compact.

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By the conjugacy theorem, there is m in R^n such that $m\,\mathrm{Cl}(f(H))m^{-1}\subset K$. In particular, we have

$$m\pi(h)\varphi(h)m^{-1} = \pi(h)(m^h\varphi(h)m^{-1}) \in K,$$

where $m^h=\pi(h)^{-1}m\pi(h)$ and $h\in H$. Hence $m^h\varphi(h)m^{-1}=1$ and in additive notation, we get $\varphi(h)=m-m^h$, $h\in H$. Therefore φ is a coboundary.

LEMMA 2. *Let G be a separable locally compact group with property* (T). *Then there exist a positive number ε and a compact subset K of G with the following condition: If π is a separable unitary representation of G on a Hilbert space $H(\pi)$ and $x\in H(\pi)$ such that $\|x\|=1$ and $|(gx,x)-1|<\varepsilon$ for all $g\in K$, then $\pi\geqq I$.*

PROOF. Suppose the assertion to be false. Let K_n be an increasing sequence of compact subsets of G such that $\bigcup_n K_n=G$. Then there are separable unitary representations of G, $\pi_1,\pi_2,\cdots$ for which there exist $x_n\in H(\pi_n)$ with $\|x_n\|=1$, $|(gx_n,x_n)-1|<1/n$ for $g\in K_n$ and $\pi_n\not\geqq I$ $(n=1,2,\cdots)$. Then consider the representation $\pi=\bigoplus_{i=1}^{\infty}\pi_i$. Due to our construction, I is contained in the closure of $\{\pi\}$.[2] However G has property (T), by [4], [6], $I\leqq\pi$ which implies $I\leqq\pi_n$ for some n. Obviously this is a contradiction.

LEMMA 3. *Let G, ε, K be described as in Lemma 2. Let π be a separable unitary representation of G in the Hilbert space $H(\pi)$. If $x\in H(\pi)$ with $\|x\|=1$ and $|(gx,x)-1|<\varepsilon^2$ for $g\in K$, then $\|gx-x\|<2\sqrt{\varepsilon}$ for all $g\in G$.*

PROOF. Let us write H_1 for the set $\{y\in H(\pi): gy=y$ for all g in $G\}$ and $H_2=H_1^{\perp}$. Clearly H_2 is invariant under G, and $H(\pi)=H_1\oplus H_2$. It is also easy to see that $\pi|_{H_2}\not\geqq I$. Let us write $x=x_1+x_2$ with $x_1\in H_1$ and $x_2\in H_2$. Since $gx=x_1+gx_2$ and $\|x\|=1$,

$$|(gx,x)-1| = |(gx_2,x_2)-(x_2,x_2)| < \varepsilon^2 \quad \text{for } g\in K.$$

Since $\pi|_{H_2}\not\geqq I$ and by Lemma 2, we must have $\|x_2\|^2<\varepsilon$. From this, $\|gx-x\|=\|gx_2-x_2\|<2\sqrt{\varepsilon}$, for all $g\in G$.

LEMMA 4. *Let G be a separable locally compact group with property* (T), *$\pi: G\to O_n$ a continuous homomorphism and $\varphi: G\to R^n$ a continuous 1-cocycle. Then $\varphi(G)$ is bounded in R^n.*

PROOF. Let $\varphi: G\to R^n$ be a continuous 1-cocycle. We define $\alpha_\lambda: G\to G\cdot R^n$ (semidirect product) by $g\to g(\lambda\varphi(g))$ $(g\in G)$. Clearly the maps

[2] Let $\tilde{G}$ be the set of all equivalence classes of separable unitary representations of G. We give $\tilde{G}$ the inner hull-kernel topology.

α_λ $(0 \leqq \lambda \leqq 1)$ are continuous homomorphisms and, as $\lambda \to 0$, $\alpha_\lambda \to$ the inclusion map i_G of G in $G \cdot R^n$. Now consider the space $L^2(G \backslash G \cdot R^n) = L^2(R^n)$. Through right translations, $G \cdot R^n$ acts unitarily on $L^2(G \backslash G \cdot R^n)$. Let B be a unit volume ball in R^n with center at 0, and x_B the characteristic function on B. Since B is invariant under O_n, $gx_B = x_B$ for all $g \in G$. Now let $\varepsilon < \frac{1}{2}$, and G, K be described as in Lemma 2. Since $\alpha_\lambda \to i_G$ as $\lambda \to 0$, there is $\delta > 0$ such that

$$|(\alpha_\lambda(g)x_B, x_B) - 1| < \varepsilon^2, \qquad g \in K,$$

and $0 \leqq \lambda \leqq \delta$. By Lemma 3, $\|\alpha_\lambda(g)x_B - x_B\| < 2\sqrt{\varepsilon}$ for all $g \in G$ and $0 \leqq \lambda \leqq \delta$. By an easy computation, $\alpha_\lambda(g)x_B = x_{B^1}$, the characteristic function on B^1 where B^1 is the translation of B by a vector $-\lambda\varphi(g)^{g^{-1}}$. Note $-\lambda\varphi(g)^{g^{-1}} = -\lambda\pi(g^{-1})(\varphi(g))$ has norm $\|\lambda\varphi(g)\|$. If $\|\lambda\varphi(g)\| \geqq 1$, $B \cap B^1 = \varnothing$, hence $\|\alpha_\lambda(g)x_B - x_B\| = \sqrt{2} > 2\sqrt{\varepsilon}$. However

$$\|\alpha_\lambda(g)x_B - x_B\| < 2\sqrt{\varepsilon} \quad \text{for all } g \in G$$

and $0 \leqq \lambda \leqq \delta$. It follows that $\|\delta\varphi(g)\| < 1$ for all $g \in G$. Hence $\varphi(G)$ is bounded in R^n.

As a consequence of Lemmas 1 and 4, we now have our main result.

THEOREM A. *Let G be a separable locally compact group with property* (T) *and $\pi: G \to O_n$ a continuous representation. Then $H^1(G, \pi) = 0$.*

As an application of Theorem A, we have the following vanishing cohomology theorem of certain discrete subgroups.

THEOREM B. *Let k be a nondiscrete locally compact field of* $\mathrm{ch}(k) = 0$, *G an affine semisimple algebraic group defined over k and Γ a discrete subgroup of $G(k)$. If the k-rank of each k-factor of $G \geqq 2$ and $G(k)/\Gamma$ has a finite Haar measure, then $H^1(\Gamma, \pi) = 0$ for every finite dimensional unitary representation π of Γ.*

PROOF. By [4], Γ has property (T).

Theorem B generalizes slightly Theorem 6.5 in [3]. However our method cannot be carried out in higher dimensional cohomology groups. In the following, we present a weak rigidity theorem for discrete groups with property (T).

THEOREM C. *Let Γ be a finitely generated discrete group with property* (T), *G a compact Lie group, and $\varphi_\lambda: \Gamma \to G$ $(0 \leqq \lambda \leqq 1)$ a continuous curve of homomorphisms. Then for each λ, there is $g_\lambda \in G$ such that $\varphi_\lambda(\gamma) = g_\lambda \varphi_0(\gamma) g_\lambda^{-1}$ for all $\gamma \in \Gamma$.*

PROOF. Let n be a positive integer. From [7], we know that there are only finitely many classes of irreducible unitary representations of Γ

with dimension $\leqq n$. Let $N_\lambda = \text{kernel}(\varphi_\lambda)$. Then the set of kernels $\{N_\lambda : 0 \leqq \lambda \leqq 1\}$ is finite. Let us denote this finite set by $\{M_1, \cdots, M_l\}$. Let $I_i = \{\lambda : N_\lambda = M_i\}$. We want to show that I_i, $(i=1, \cdots, l)$ are closed subsets of $[0, 1]$. Let us write Γ_i for Γ/M_i. Clearly if $\lambda \in \text{Cl}(I_i)$, $N_\lambda \supset M_i$. Hence for each $\lambda \in \text{Cl}(I_i)$, we set $\bar{\varphi}_\lambda$ for the homomorphism of Γ/M_i induced by φ_λ. Let $\mathscr{R}(\Gamma_i, G)$ be the space of all representations of Γ_i in G. We equip $\mathscr{R}(\Gamma_i, G)$ with the compact open topology. G acts continuously on $\mathscr{R}(\Gamma_i, G)$ by $(g \circ \varphi)(\gamma) = g\varphi(\gamma)g^{-1}$, $\varphi \in \mathscr{R}(\Gamma_i, G)$, $\gamma \in \Gamma_i$ and $g \in G$. By Theorem A and [8], $G \circ \bar{\varphi}_\lambda$, $(\lambda \in I_i)$ are open subsets of $\mathscr{R}(\Gamma_i, G)$. Since G is compact, $G \circ \bar{\varphi}_\lambda$, $(\lambda \in I_i)$ are compact-open in $\mathscr{R}(\Gamma_i, G)$. On the other hand $\mathscr{R}(\Gamma_i, G)$ can be viewed as an R-algebraic variety. It follows that $\mathscr{R}(\Gamma_i, G)$ has only finitely many arcwise connected components [9]. Hence

$$\bigcup_{\lambda \in I_i} G \circ \bar{\varphi}_\lambda = \bigcup_{j=1}^{m} G \circ \bar{\varphi}_{\lambda_j}$$

for some finitely many elements $\lambda_1, \cdots, \lambda_m$ in I_i, and consequently $\bigcup_{\lambda \in I_i} G \circ \bar{\varphi}_\lambda$ is compact-open in $\mathscr{R}(\Gamma_i, G)$. Therefore $\text{Cl}(I_i) = I_i$ $(i = 1, \cdots, l)$ and l has to be 1. Again by Theorem A and [8] $G \circ \bar{\varphi}_0$ is compact-open in $\mathscr{R}(\Gamma_1, G)$. Since $\{\bar{\varphi}_\lambda : 0 \leqq \lambda \leqq 1\}$ is connected, one concludes readily that $\bar{\varphi}_\lambda \in G \circ \bar{\varphi}_0$ for all λ. Therefore $\varphi_\lambda \in G \circ \varphi_0$ for all λ.

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