

MEROMORPHIC CONTINUATION OF THE  $\mathcal{S}$ -MATRIX  
FOR THE OPERATOR  $-\Delta$  ACTING  
IN A CYLINDER

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**ABSTRACT.** Let  $A^e$  and  $A$  denote the selfadjoint operators given by  $-\Delta$  associated with zero boundary conditions in the domains  $S$  and  $\Omega$ , respectively, where  $S$  is a semi-infinite (or infinite) cylinder with arbitrary cross-section in  $N$ -dimensional Euclidean space ( $N \geq 2$ ) and  $\Omega$  is obtained from  $S$  by perturbing a finite portion of the boundary of  $S$ . It has been previously shown that there exists a set of intervals,  $G_m = [\nu_m, \nu_{m+1})$ ,  $m=1, 2, \dots$ , such that  $0 < \nu_m < \nu_{m+1} < \infty$ ,  $A_0$  has spectral multiplicity  $m$  on  $G_m$  and there is a unitary  $\mathcal{S}$ -matrix,  $\mathcal{S}_m(\lambda)$ , of rank  $m$  corresponding to each  $G_m$ , whose elements may be explicitly given. It is now shown that  $\mathcal{S}_m(\lambda)$  may be meromorphically continued onto the Riemann surface  $R_m$ , obtained by making each  $\nu_j$  a branch point of order one,  $j=1, \dots, m$ . Furthermore, the poles are shown to correspond to resonant states.

**1. Introduction.** The purpose of this paper is to consider the singularities of the  $\mathcal{S}$ -matrix associated with the perturbation and scattering problem discussed in [1]–[3]. This situation differs considerably from quantum mechanical scattering in that here there occurs an infinite number of “nonphysical” sheets and the rank of the  $\mathcal{S}$ -matrix varies with the spectral parameter.<sup>2</sup> In this section we shall first briefly recall the relevant results obtained in [1]–[3] and then outline the results of the remainder of this paper.

Let  $S$  denote the semi-infinite cylinder in  $R^N$ ,  $N$ -dimensional Euclidean space ( $N \geq 2$ ), with arbitrary bounded cross-section  $l$ . Thus,  $S$  consists of the points  $x = (x_1, \dots, x_N) = (\tilde{x}, x_N) \in R^N$ , where  $\tilde{x} \in l$  and  $x_N \geq 0$ .<sup>3</sup>

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<sup>2</sup> The results of this paper are closely connected to the problem of propagation of waves in a waveguide.

<sup>3</sup> We might just as easily consider the infinite cylinder  $S'$  (i.e.,  $\tilde{x} \in l$ ,  $x_N \in R^1$ ).

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Now we perturb a finite portion of  $\dot{S}$ , the boundary of  $S$ , to obtain a semi-infinite domain  $\Omega$  with a sufficiently smooth boundary  $\dot{\Omega}$ ,<sup>4</sup> such that  $\Omega=S$  for each point  $x=(\bar{x}, x_N)$  for which  $x_N$  is sufficiently large, say  $x_N \geq \bar{x}_N$ .

Let  $A_0(A)$  denote the selfadjoint operator acting in the Hilbert space  $H_0=L_2(S)$  ( $H=L_2(\Omega)$ ), given by  $-\Delta=-\sum_{j=1}^N(\partial^2/\partial x_j^2)$  associated with the zero Dirichlet boundary condition in the domain  $S(\Omega)$ . Let  $A_l$  denote the corresponding operator defined in  $L_2(l)$  and let  $\{v_n\}$  and  $\eta_n(\bar{x})$  denote a complete set of eigenvalues (in increasing order) and corresponding orthonormal eigenfunctions for  $A_l$ .

Let  $A^c$  denote the continuous part of  $A$  (i.e. that part of  $A$  orthogonal to all of its eigenfunctions). Let  $\Lambda$  denote the set of eigenvalues of  $A$ . It was shown in [1] that a complete set of generalized eigenfunctions for  $A_0$  is given by

$$w_n^0(x; \lambda) = (2/\pi)^{1/2} \sin(\lambda - v_n)^{1/2} x_N \eta_n(\bar{x})$$

for each  $\lambda \in (v_m, v_{m+1})$ ,  $n=1, \dots, m$ , and arbitrary  $m$ . It was also shown in [1] employing the limiting absorption principle that two complete sets of generalized eigenfunctions,  $w_n^\pm(x; \lambda) = w_n^0(x; \lambda) \pm v_n^\pm(x; \lambda)$ , may be constructed for  $A^c$ , where  $\lambda \in (v_m, v_{m+1}) - \Lambda$ ,  $n=1, 2, \dots, m$ , and  $m$  is arbitrary.

The functions  $v_n^-(x; \lambda)$  satisfy the following incoming radiation condition for  $x_N \geq \bar{x}_N$ :

$$(1.1) \quad v_n^-(x; \lambda) = \sum_{n'=1}^m c_{n'}^{n-}(\lambda) \exp\{-i(\lambda - v_{n'})^{1/2} x_N\} \eta_{n'}(\bar{x}) \\ + \sum_{n'=m+1}^{\infty} c_{n'}^{n-}(\lambda) \exp\{-(v_{n'} - \lambda)^{1/2} x_N\} \eta_{n'}(\bar{x}),$$

where  $(\ )^{1/2}$  denotes the positive square root and  $\lambda \in (v_m, v_{m+1}) - \Lambda$ . The outgoing radiation condition for  $v_n^+(x; \lambda)$  is defined similarly with  $(\lambda - v_{n'})^{1/2}$  replaced by  $-(\lambda - v_{n'})^{1/2}$ ,  $n'=1, \dots, m$ , and  $c_{n'}^{n-}(\lambda)$  by  $c_{n'}^{n+}(\lambda)$ .

It was also shown in [1] that wave operators  $W^\pm$  may be constructed by stationary methods in terms of the functions  $w_n^0(x; \lambda)$  and  $w_n^\pm(x; \lambda)$ . In [2],  $W^\pm$  were expressed by means of a time-dependent approach and the scattering operator  $\mathcal{S}$  was defined by  $\mathcal{S} = W^{+-1} W^-$ .

The present paper deals with the  $\mathcal{S}$ -matrices  $\mathcal{S}_m(\lambda)$  which were shown in [3] to yield a convenient representation for  $\mathcal{S}$ . It was proved in [3] that the operator  $\mathcal{S}_m(\lambda)$ , acting in  $C^m$ , the  $m$ -dimensional complex

<sup>4</sup> Given an arbitrary domain  $D$ , by  $\dot{D}$  we shall mean its boundary.

unitary space, is given by

$$(1.2) \quad \mathcal{S}_m(\lambda) = I + T_m(\lambda),$$

where  $\lambda \in (\nu_m, \nu_{m+1}) - \Lambda$ ,  $I$  denotes the identity operator and  $T_m(\lambda)$  is represented by the following matrix:

$$(1.3) \quad T_m(\lambda) = (t_{n,n'}(\lambda)), \quad n, n' = 1, 2, \dots, m,$$

with  $t_{n,n}(\lambda) = -(2\pi)^{1/2} c_n^-(\lambda)$ .

$\mathcal{S}_m(\lambda)$  is a unitary mapping on  $C^m$ .

An interesting feature of this result is the fact that the rank  $m$  of the matrix  $\mathcal{S}_m(\lambda)$  varies with  $\lambda$ . This is a consequence of the fact that  $m_0(\lambda) = m$  for each  $\lambda \in [\nu_m, \nu_{m+1})$ , where  $m_0(\lambda)$  denotes the spectral multiplicity of  $A_0$  at the point  $\lambda$ . Hence a new "channel" is added to the spectrum at each point  $\nu_m$ . It is for this reason that our results differ considerably from those obtained in [4] and [5]. Furthermore, we shall not need to employ the integral equation method of [5] or the techniques or results developed in [4].

In §2, we shall meromorphically continue  $\mathcal{S}_m(\lambda)$  to certain "non-physical sheets". In §3, we shall relate the poles of  $\mathcal{S}_m(\lambda)$  to "resonant states". We shall see that there are  $2^m - 1$  nonphysical sheets associated with  $\mathcal{S}_m(\lambda)$ , whereas in [4] and [5] there is only one nonphysical sheet associated with the  $\mathcal{S}$ -matrix.

**2. Poles of the  $\mathcal{S}$ -matrix.** In order to deal with the  $\mathcal{S}$ -matrix, it will be convenient to restrict our operators  $A_0$  and  $A$  to closed intervals  $\mathcal{G} = [a, b] \subseteq \sigma(A^e)$  over which the  $\mathcal{S}$ -matrix has constant rank. Hence we assume that  $A$  has no eigenvalues in  $\mathcal{G}$  and that  $\mathcal{G} \subset (\nu_m, \nu_{m+1})$  for some fixed integer  $m$ . Set  $A_{0\mathcal{G}} = A_0 E_{0\mathcal{G}}$  ( $A_{\mathcal{G}} = A E_{\mathcal{G}}$ ), where  $\{E_{0\lambda}\}$  ( $\{E_{\lambda}\}$ ) denotes the spectral resolution of  $A_0$  ( $A$ ). We are thus interested in the scattering operator,  $\mathcal{S}_{\mathcal{G}} = \mathcal{S} E_{0\mathcal{G}}$ , associated with  $A_{0\mathcal{G}}$  and  $A_{\mathcal{G}}$ . Our aim in this section is to meromorphically continue the  $\mathcal{S}$ -matrix  $\mathcal{S}_m(\lambda)$ , defined by (1.2) and (1.3) and associated with  $\mathcal{S}_{\mathcal{G}}$ , to values of  $\lambda$  outside of  $\mathcal{G}$ .

We observe from (1.1) that the function  $v_n^-(x; \lambda)$  has a branch point of order one at each point  $\lambda = \nu_j$ ,  $j = 1, 2, \dots$ . Let us denote by  $R_{\infty}$  the infinitely sheeted Riemann surface obtained by making each point  $\nu_j$  a branch point of order one,  $j = 1, 2, \dots$ . By  $\Gamma_{n_1, n_2, \dots, n_k}$  ( $\text{Cl}(\Gamma_{n_1, n_2, \dots, n_k})$ ), we shall mean that open (closed) sheet of  $R_{\infty}$  consisting of those points  $\lambda$  for which

$$0 < \arg(\lambda - \nu_j) < 2\pi \quad (0 \leq \arg(\lambda - \nu_j) < 2\pi), \quad j = n_1, \dots, n_k,$$

and

$$-2\pi < \arg(\lambda - \nu_j) < 0 \quad (-2\pi \leq \arg(\lambda - \nu_j) < 0) \quad \text{for all remaining } j,$$

where  $n_1, \dots, n_k$  are positive integers. Thus we may proceed from the sheet  $\Gamma_{n_1, \dots, n_{k-1}}$  to  $\Gamma_{n_1, \dots, n_{k-1}, n_k}$  by traversing the point  $v_{n_k}$  once by means of a curve that does not traverse any of the remaining  $v_j$ . The "physical sheet",  $\Gamma_0$  ( $\Gamma_0$ ), shall consist of those  $\lambda$  satisfying  $-2\pi < \arg(\lambda - v_j) < 0$  ( $-2\pi \leq \arg(\lambda - v_j) < 0$ ),  $j=1, 2, \dots$ . By  $R_m$ , we shall mean that part of  $R_\infty$  consisting of the sheets  $\Gamma_0$  and  $\text{Cl}(\Gamma_{n_1, \dots, n_k})$  for all integers  $n_1, \dots, n_k$  such that  $1 \leq n_1, \dots, n_k \leq m$ .  $R_m$  clearly has  $2^m$  distinct sheets.

We shall first define  $\mathcal{S}_m(\lambda)$  on  $\Gamma_0$  and then meromorphically continue  $\mathcal{S}_m(\lambda)$  to the remainder of  $R_m$ . Now suppose  $\lambda \in \Gamma_0$ . Hence  $\text{Im}(\lambda - v_j)^{1/2} < 0, j=1, 2, \dots$ . Employing the operator  $(A - \lambda)^{-1}$ , we obtained functions  $v_n^-(x; \lambda)$  in [1] such that

$$(2.1) \quad v_n^-(x; \lambda) = \sum_{n'=1}^{\infty} c_{n'}^-(\lambda) \exp\{-i(\lambda - v_{n'})^{1/2} x_N\} \eta_{n'}(\tilde{x}) \quad \text{for } x_N \geq \tilde{x}_N$$

and such that  $v_n^-(x; \lambda)$  satisfies  $(\Delta + \lambda)v_n^-(x; \lambda) = 0$  in  $\Omega$ , and  $v_n^-(x; \lambda) = -w_n^0(x; \lambda)$  on  $\tilde{\Omega}, n=1, 2, \dots$ , for each  $\lambda \in \Gamma_0 - \Lambda$  (i.e.  $\lambda$  is not an eigenvalue of  $A$  below  $v_1$ ). We now define  $\mathcal{S}_m(\lambda)$  for  $\lambda \in \Gamma_0 - \Lambda$  by (1.2) and (1.3), where

$$c_n^-(\lambda) = \exp\{i(\lambda - v_n)^{1/2} x_N\} \int_{l_{x_N}} v_n^-(x; \lambda) \eta_n(\tilde{x}) d\tilde{x}$$

is defined by (2.1).  $(A - \lambda)^{-1}$  is clearly a meromorphic function of  $\lambda$  in  $\Gamma_0$  with poles occurring at the points  $\lambda \in \Lambda \cap (-\infty, v_1)$ . Hence the only singularities of  $\mathcal{S}_m(\lambda)$  are at worst poles at these points.

We next define  $\mathcal{S}_m(\lambda)$  on the first nonphysical sheet  $\Gamma_1$ . To this end, set  $\kappa = (\lambda - v_1)^{1/2}$  and  $\mathcal{S}_m(\kappa) = \mathcal{S}_m(\lambda)$  for each  $\lambda \in \Gamma_0 - \Lambda$ . Thus  $\text{Im } \kappa < 0$ . We wish to meromorphically continue  $\mathcal{S}_m(\kappa)$  from the lower half of the  $\kappa$ -plane into the upper half plane across the interval  $I_{\mathcal{G}} = [(a - v_1)^{1/2}, (b - v_1)^{1/2}]$ . Let us bear in mind that all our values of  $\kappa$  are chosen so that

$$(2.2) \quad \text{Im}(\kappa^2 + v_1 - v_j)^{1/2} = \text{Im}(\lambda - v_j)^{1/2} < 0, \quad j = 2, \dots.$$

Thus our aim is to show that  $\mathcal{S}_m(\lambda)$  has a meromorphic continuation from  $\Gamma_0$  to  $\Gamma_1$  across that part of the branch cut in the  $\lambda$ -plane contained in  $\mathcal{G}$ .

To prove this, we shall need the following lemma.

**LEMMA 2.1.** *Suppose that  $L(\kappa)$  is a compact, linear operator acting in a Banach space  $B$  for each  $\kappa$  in a domain  $D$  and that  $L(\kappa)$  is a meromorphic function of  $\kappa$  in the operator topology on  $B$  for each  $\kappa$  in  $D$ . Then either  $I + L(\kappa)$  is invertible for no  $\kappa$  in  $D$  or else  $M(\kappa) = (I + L(\kappa))^{-1}$  exists and is analytic everywhere in  $D$  except at a discrete set of points, corresponding*

to poles of  $M(\kappa)$ , provided we have either

- (a)  $L(\kappa)$  is analytic in  $D$ , or
- (b)  $B$  is finite dimensional.

The proof of (a) was given in both [6] and [7] and the proof of (b) follows from [7, Corollary II]. Now suppose that  $\text{Im } \kappa > 0$  and (2.2) holds. We set

$$(2.3) \quad \tilde{\mathcal{S}}_m(\kappa) = \mathcal{S}_m(\bar{\kappa})^{*-1},$$

where  $\mathcal{S}_m(\kappa)^*$  denotes the adjoint of  $\mathcal{S}_m(\kappa)$  and (2.2) now holds with  $\kappa$  replaced by  $\bar{\kappa}$  and  $\lambda$  replaced by  $\bar{\lambda} \in \Gamma_0 - \Lambda$  ( $\bar{\lambda} = \bar{\kappa}^2 + \nu_1$ ). We readily conclude from Lemma 2.1(b) and our previous discussion of  $\mathcal{S}_m(\lambda)$  that  $\tilde{\mathcal{S}}_m(\kappa)$  is a meromorphic function of  $\kappa$  in  $\text{Im } \kappa > 0$ . If we set  $\tilde{\mathcal{S}}_m(\kappa) = \mathcal{S}_m(\lambda)$  for each  $\lambda = \kappa^2 + \nu_1 \in \mathcal{G}$ , we see from the unitarity of  $\mathcal{S}_m(\lambda)$  that (2.3) also holds in this case. (We recall from (1.1) that  $\arg(\lambda - \nu_j)^{1/2} = 0$ ,  $j = 1, \dots, m$ . Hence we assume that  $\arg \kappa = 0$  for  $\kappa \in I_{\mathcal{G}}$ .) Finally we may immediately see from the proof of the limiting absorption principle in [1, (2.3)], and the unitarity of  $\mathcal{S}_m(\kappa)$  (for  $\kappa \in I_{\mathcal{G}}$ ) that  $\tilde{c}_{n'}^{n'}(\kappa) (= c_{n'}^{n'}(\lambda))$ ,  $n, n' = 1, \dots, m$ ,  $\mathcal{S}_m(\kappa)$  and hence  $\mathcal{S}_m(\kappa)^{*-1}$  are continuous functions of  $\kappa$  for each  $\kappa$  in the closure of a sufficiently small neighborhood of  $I_{\mathcal{G}}$  contained in the lower half plane. Therefore we conclude from these facts and the Cauchy integral formula as in the proof of the Schwarz reflection principle that (2.3) yields our desired meromorphic continuation into the upper half plane. (In fact  $\mathcal{S}_m(\kappa)$  is analytic for  $\kappa$  in a neighborhood of  $I_{\mathcal{G}}$ .)

Using analogous arguments, we may continue  $\mathcal{S}_m(\lambda)$  meromorphically across  $\mathcal{G}$  from  $\Gamma_0$  onto  $\Gamma_j$ ,  $j = 1, \dots, m$  (in the sense described above). Starting with these continued values of  $\mathcal{S}_m(\lambda)$ , we may then repeat the above arguments to obtain meromorphic continuations of  $\mathcal{S}_m(\lambda)$  onto the sheets  $\Gamma_{ij}$ ,  $i, j = 1, 2, \dots, m$ . Using an induction argument, we finally obtain

**THEOREM 2.1.** *The  $\mathcal{S}$ -matrix  $\mathcal{S}_m(\lambda)$ , defined by (1.2) and (1.3) for each  $\lambda \in \mathcal{G}$ , has a meromorphic continuation across  $\mathcal{G}$  (in the sense described above) to each sheet  $\Gamma_0, \Gamma_{n_1, \dots, n_k}$ ,  $1 \leq n_1, \dots, n_k \leq m$ , of the Riemann surface  $R_m$ .*

**REMARK 2.1.** We may obtain analogous results for the interval  $[-(b - \nu_j)^{1/2}, -(a - \nu_j)^{1/2}]$  in the  $\kappa$ -plane using the same arguments.

**REMARK 2.2.** We are not able to employ this method to study the continuation of  $\mathcal{S}_m(\lambda)$  across  $\mathcal{G}$  onto any of the sheets  $\Gamma_j$ ,  $j > m$ , for the following reason. If we set  $\kappa = (\lambda - \nu_j)^{1/2}$ , then  $\kappa$  is purely imaginary and  $\text{Im } \kappa < 0$  for  $\lambda \in \mathcal{G} \cap \Gamma_0$ . Hence  $\tilde{\mathcal{S}}_m(\kappa)$  is not unitary on a portion of the boundary of the lower half plane but rather on an interior set of points.

Therefore, we cannot employ the reflection principle to meromorphically continue  $\mathcal{S}_m(\kappa)$  onto the half plane  $\text{Im } \kappa > 0$  (i.e., to continue  $\mathcal{S}_m(\lambda)$  onto  $\Gamma_j$ ).

**3. The resonant states.** In this section we obtain an interesting characterization of the poles of  $\mathcal{S}_m(\lambda)$  on  $R_m$ . We shall say that  $\lambda_0 \in R_m$  is a pole of  $\mathcal{S}_m(\lambda)$  if  $\kappa_0 = (\lambda_0 - \nu_{n_j})^{1/2}$  is a pole of  $\tilde{\mathcal{S}}_m(\kappa^2 + \nu_{n_j})$ ,  $n_j \leq m$ , corresponding to any one of the continuations of Theorem 2.1.

**DEFINITION 3.1.** Suppose that  $m$  is a fixed integer,  $\lambda_0 \in \Gamma_j$ ,  $1 \leq j \leq m$ , and there exists a nontrivial solution,  $w(x; \bar{\lambda}_0)$ , of the boundary value problem

$$(3.1) \quad (\Delta + \bar{\lambda}_0)w(x; \bar{\lambda}_0) = 0 \text{ in } \Omega, \quad w(x; \bar{\lambda}_0) = 0 \text{ on } \Omega,$$

where  $\bar{\lambda}_0$  denotes that value of  $\bar{\lambda}_0 \in \Gamma_0$ . Suppose also that there exist constants  $c_n, n = 1, 2, \dots$ , with some  $c_J \neq 0, 1 \leq J \leq m$ , such that

$$(3.2) \quad \begin{aligned} w(x; \bar{\lambda}_0) = & \sum_{n=1}^m c_n \exp\{i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\tilde{x}) \\ & + \sum_{n=m+1}^{\infty} c_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\tilde{x}) \end{aligned}$$

for  $x_N \geq \hat{x}_N$ . Then we shall say that  $\lambda_0$  is a  $\Gamma_j$  resonant state for  $A_{\mathcal{G}}$  ( $\mathcal{G}$  as in Theorem 2.1).

*Note.* The function  $w(x; \bar{\lambda}_0)$  is exponentially blowing up for  $x_N$  large.

**THEOREM 3.1.** Suppose that the interval  $\mathcal{G}$  is the same as in Theorem 2.1,  $\lambda_0 \in \Gamma_j, 1 \leq j \leq m$ , and  $\bar{\lambda}_0 \in \Gamma_0 - \Lambda$ . Then we have the following results.

(a) If  $\lambda_0$  is a pole of  $\mathcal{S}_m(\lambda)$ , then  $\lambda_0$  is a  $\Gamma_j$  resonant state for  $A_{\mathcal{G}}$  and the constants  $c_n, n = 1, \dots, m$ , in (3.2) are expressed in terms of the matrix  $\mathcal{S}_m(\bar{\lambda}_0)^*$  by (3.3) and (3.4) below.

(b) If  $\lambda_0$  is a  $\Gamma_j$  resonant state for  $A_{\mathcal{G}}$ , then  $\lambda_0$  is a pole of  $\mathcal{S}_m(\lambda)$ .

**PROOF.** (a) It follows from the hypotheses and our previous discussion of  $\mathcal{S}_m(\lambda)$  that  $\mathcal{S}_m(\bar{\lambda}_0)^*$  is not invertible. Hence there exists a nontrivial  $h = (h_1, \dots, h_m) \in C^m$  such that  $\mathcal{S}_m(\bar{\lambda}_0)^* h = 0$ , or equivalently

$$(3.3) \quad (2\pi)^{1/2} i \sum_{n'=1}^m \bar{c}_{n'}(\bar{\lambda}_0) h_{n'} = -h_n, \quad n = 1, \dots, m.$$

Set

$$\begin{aligned} w(x; \bar{\lambda}_0) &= \sum_{n=1}^m \bar{h}_n w_n^-(x; \bar{\lambda}_0) \\ &= \sum_{n=1}^m \bar{h}_n w_n^0(x; \bar{\lambda}_0) \\ &\quad + \sum_{n=1}^m \bar{h}_n \sum_{n'=1}^{\infty} c_{n'}^-(\bar{\lambda}_0) \exp\{-i(\bar{\lambda}_0 - \nu_{n'})^{1/2} x_N\} \eta_{n'}(\tilde{x}) \quad (x_N \geq \hat{x}_N). \end{aligned}$$

Since

$$w_n^0(x; \bar{\lambda}_0) = \frac{1}{2i} \left(\frac{2}{\pi}\right)^{1/2} (\exp\{i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} - \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\}) \eta_n(\bar{x}),$$

we see from (3.3) that

$$w(x; \bar{\lambda}_0) = \sum_{n=1}^m \frac{\bar{h}_n}{(2\pi)^{1/2} i} \exp\{i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}) + \sum_{n=m+1}^{\infty} c_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}), \quad x_N \geq \dot{x}_N,$$

for some constants  $c_n, n=m+1, \dots$ . Thus  $w(x; \bar{\lambda}_0)$  clearly satisfies (3.1) and (3.2) and is nontrivial (since  $h \neq 0$ ). This proves (a).

(b) Suppose there exists a solution of (3.1) that satisfies (3.2) with at least one nonzero  $c_n, 1 \leq n \leq m$ . Set

$$(3.4) \quad h_n = -(2\pi)^{1/2} i \bar{c}_n, \quad n = 1, \dots, m.$$

It follows readily that

$$(3.5) \quad \begin{aligned} w(x; \bar{\lambda}_0) &= \sum_{n=1}^m \frac{1}{(2\pi)^{1/2} i} \bar{h}_n \exp\{i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}) \\ &\quad + \sum_{n=m+1}^{\infty} c_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}) \\ &= \sum_{n=1}^m \bar{h}_n w_n^0(x; \bar{\lambda}_0) + \sum_{n=1}^m \frac{1}{(2\pi)^{1/2} i} \bar{h}_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}) \\ &\quad + \sum_{n=m+1}^{\infty} c_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}), \quad x_N \geq \dot{x}_N. \end{aligned}$$

But we have

$$(3.6) \quad \begin{aligned} \sum_{n=1}^m \bar{h}_n w_n^-(x; \bar{\lambda}_0) &= \sum_{n=1}^m \bar{h}_n w_n^0(x; \bar{\lambda}_0) \\ &\quad + \sum_{n'=1}^m \left( \sum_{n=0}^m c_{n'}^-(\bar{\lambda}_0) \bar{h}_n \right) \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}) \\ &\quad + \sum_{n=m+1}^{\infty} d_n \exp\{-i(\bar{\lambda}_0 - \nu_n)^{1/2} x_N\} \eta_n(\bar{x}), \\ &\quad x_N \geq \dot{x}_N, \text{ for some constants, } d_n, n = m + 1, \dots \end{aligned}$$

We readily see from (3.5) and (3.6) that  $w(x; \bar{\lambda}_0) - \sum_{n=1}^m \bar{h}_n w_n^-(x; \bar{\lambda}_0)$  either vanishes identically or else is an eigenfunction of  $\mathcal{A}$  corresponding to  $\bar{\lambda}_0$ . Since  $\bar{\lambda}_0 \notin \Lambda$ , we thus conclude that  $w(x; \bar{\lambda}_0) = \sum_{n=1}^m \bar{h}_n w_n^-(x; \bar{\lambda}_0)$ .

Equating coefficients in (3.5) and (3.6), we conclude that

$$h_n = -(2\pi)^{1/2} i \sum_{n'=1}^m \bar{c}_{n'}(\bar{\lambda}_0) h_{n'}, \quad n = 1, \dots, m.$$

Hence  $\mathcal{S}_m(\bar{\lambda}_0)^* h = 0$ ,  $h = (h_1, \dots, h_m) \neq 0$ , so that  $\lambda_0$  is a pole of  $\mathcal{S}_m(\lambda)$ .  
Q.E.D.

REMARK 3.1. A result analogous to Theorem 3.1 may be proved for an arbitrary sheet,  $\Gamma_{n_1, \dots, n_k}$  of  $R_m$  using similar reasoning. However, it is necessary first to meromorphically continue the functions  $w_n^-(x; \bar{\lambda}_0)$  to  $\Gamma_{n_1, \dots, n_{k-1}}$ . Hence we postpone this result to a future publication.

REMARK 3.2. The results of this paper hold even if the coefficients and boundary conditions are perturbed, provided the perturbation has bounded support.  $A$  may even be nonselfadjoint. The arguments remain essentially unchanged.

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