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CONTINUED FRACTIONS AND EQUIVALENT COMPLEX NUMBERS

RICHARD B. LAKEIN

ABSTRACT. In this note it is shown, by a counterexample, that the familiar theorem on the continued fraction expansions of equivalent numbers does not hold when these notions are extended to complex numbers.

Two real numbers x, x' are called *equivalent*, $x \sim x'$, if

(1) $x' = (ax + b)/(cx + d)$ for some $a, b, c, d \in Z$, $ad - bc = \pm 1$.

Consider also the continued fraction (CF) expansion of a real number

(2) $x = (a_0, a_1, \cdots, a_{n-1}, x_n), \qquad x_{n-1} = a_{n-1} + 1/x_n,$

where x_n is the nth *complete quotient* of x. It is a standard theorem in CF's that $x \sim x'$ if and only if, in the CF expansions of x and x', there exist m, n such that $a_{m+k} = a'_{n+k}$ for all $k \geqq 0$—more briefly, $x_m = x'_n$.

Hurwitz, in a paper [2] on the "nearest integer" CF (where the partial quotients a_n may be negative) proved that essentially the same result carries over. That is, $x \sim x'$ if and only if there exist m, n such that $x_m = \pm x'_n$, where these are complete quotients of the nearest integer CF's. Hurwitz also defined [1] a complex generalization of the nearest integer CF (it might be called the "nearest Gaussian integer" CF) by which a complex number x is expanded in a simple CF as in (2) with partial quotients a_n in $Z[i]$. There is an analogous notion of equivalent numbers as in (1), where $a, b, c, d \in Z[i]$, $ad - bc = \pm 1, \pm i$.

Although this complex CF has many analogies to real CF's, the expected theorem on equivalent numbers fails, as shown by a

COUNTEREXAMPLE. Let $\Omega = \frac{1}{2}(i + (43 + 28i)^{1/2})$, $A = (5 - i + \Omega)/(4 - i)$, $B = (3 + 2i + \Omega)/4$. Then $A \sim B$, in fact $A = (2B - i)/(B - i)$. However the CF expansions of A and B, which are periodic, are distinct:

$$A = \overline{(2 + i, 3i, -1 + 2i, -1 + 2i, 3, -2 - i)},$$
$$B = \overline{(2 + i, -2 + i, -2 + i, 1 - 2i, -1 - 2i, 1 + 2i)}.$$

Thus $A_m = \pm B_n$, or even $\pm iB_n$, never holds.

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Finally we note that an incomplete partial result may hold for a different, less natural type of complex CF. See [3], [4, p. 88].

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Department of Mathematics, State Universtiy of New York at Buffalo, Amherst, New York 14226