POSITIVE CONTRACTIONS ON $L_1$-SPACES

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Abstract. Let $T$ be a positive linear operator on $L_1$ of a probability space such that $\|T\|_1 \leq 1$. In this note we consider the following question: Under what condition is $T$ multiplicative on $L_\infty$?

Let $(X, \mathcal{F}, m)$ be a probability space and $L_1 = L_1(X, \mathcal{F}, m)$ the Banach space of equivalence classes of integrable complex-valued functions on $X$. Let $T$ be a positive linear operator on $L_1$ such that $\|T\|_1 = 1$, and define $G(T) = \{f \in L_1; |f| = |Tf| = 1\}$. The main purpose of this note is to prove the following:

Theorem. If $G(T)$ is total in $L_1$, i.e., the linear manifold generated by $G(T)$ is dense in $L_1$ in the norm topology, then for any bounded functions $f$ and $g$ we have $T(fg) = (Tf)(Tg)$.

As a corollary of the Theorem we have the following result; essentially the same idea has been used by Halmos [2, p. 45] to find a necessary and sufficient condition for a unitary operator on $L_2 = L_2(X, \mathcal{F}, m)$ to be induced by an automorphism of the measure algebra $(\mathcal{F}(m), m)$ associated with $(X, \mathcal{F}, m)$.

Corollary. A necessary and sufficient condition that $T$ be induced by a homomorphism of the measure algebra $(\mathcal{F}(m), m)$ into itself is that $G(T)$ be total in $L_1$.

For the proof of the theorem we need some lemmas.

Lemma 1. Let $K$ be a compact Hausdorff space and $C(K)$ the Banach algebra of all complex-valued continuous functions $f$ on $K$ with norm $\|f\|_u = \sup\{|f(s)|; s \in K\}$. Let $U$ be a positive linear operator on $C(K)$ such that $U1 = 1$. Let $f, g \in C(K)$, $|f| = |g| = 1$, and $|Uf| = |Ug| = 1$. Then $U(fg) = (Uf)(Ug)$.

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Proof. For a fixed point \( s \) in \( K \), the linear functional \( h \mapsto (Uh)(s) \) is positive and continuous. The Riesz representation theorem implies that there exists a positive Radon measure \( m(s, dt) \) on \( K \) such that

\[
(Uh)(s) = \int h(t) m(s, dt), \quad h \in C(K).
\]

Since \( (Uf)(s) = \int f(t) m(s, dt) \) and \( |f| = |Uf| = 1 \), it follows that \( f = (Uf)(s) \) almost everywhere with respect to the measure \( m(s, dt) \). Similarly, \( g = (Ug)(s) \) almost everywhere with respect to \( m(s, dt) \). Consequently, \( U(fg)(s) = \int f(t)g(t) m(s, dt) = (Uf)(s)(Ug)(s) \), and the lemma is proved.

Applying the lemma above, we have the following

Lemma 2. Let \( |f| = |g| = 1 \) and \( |Tf| = |Tg| = 1 \). Then \( T(fg) = (Tf)(Tg) \).

Proof. Since \( 1 = |Tf| \leq |f| = 1 \) and \( \|T\|_1 \leq 1 \), we observe \( T1 = 1 \). Thus \( T \) maps \( L_1 = L_1(X, \mathcal{F}, m) \) into itself and \( \|T\|_1 \leq 1 \). The Gel'fand-Naimark theorem implies that \( L_1 \) is isometrically isomorphic with \( C(K) \) for some compact Hausdorff space \( K \). Hence we may associate with \( T \) a linear operator \( U \) on \( C(K) \) with \( U1 = 1 \). It is now easy to check that \( U \) is positive and that if \( f \) and \( g \) are the functions in \( C(K) \) corresponding to \( f \) and \( g \), respectively, then \( |f| = |g| = 1 \) and \( |Uf| = |Ug| = 1 \). Therefore \( U(fg) = (Uf)(Ug) \) by Lemma 1. This completes the proof of Lemma 2.

Proof of Theorem. Let \( M \) denote the linear manifold generated by \( T1 \), and let \( f, g \) be bounded functions. Since \( M \) is a dense subset of \( L_1 \) consisting of bounded functions, there exist \( f_n, g_n \in M \), \( n=1, 2, \ldots \), such that

\[
\lim_{n} \|f_n g_n - fg\|_1 = \lim_{n} \|f_n - f\|_1 = \lim_{n} \|g_n - g\|_1 = 0.
\]

Lemma 2 implies that \( T(f_n g_n) = (Tf_n)(Tg_n) \) for each \( n \), and hence we have \( T(fg) = (Tf)(Tg) \). This completes the proof.

Proof of Corollary. Since the necessity of the condition is obvious, we prove here only the sufficiency of it.

Suppose \( G(T) \) is total in \( L_1 \). The theorem implies that if \( A \in \mathcal{F} \) then \( (T1_A)^2 = T1_A \). This shows that \( T1_A = 1_B \) for some \( B \in \mathcal{F} \). Thus if we set \( \varphi(A) = B \), then \( \varphi \) defines a mapping of the measure algebra \( (\mathcal{F}(m), m) \) into itself. Since \( \|T\|_1 \leq 1 \) and \( T1 = 1 \), it follows that

\[
\varphi(A - B) = \varphi(A) - \varphi(B), \quad \varphi \left( \bigcup_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} \varphi(A_n)
\]

and

\[
m(A) = m(\varphi(A)),
\]

whenever \( A, B, \) and \( A_n, n=1, 2, \ldots \), are elements of \( (\mathcal{F}(m), m) \). That is
to say, \( \varphi \) is a homomorphism of \((\mathcal{F}(m), m)\) into itself, and the Corollary is proved.

Let \( K, C(K), \) and \( U \) be the same as in Lemma 1, and define \( F(U) = \{ f \in C(K); |f| = |Uf| = 1 \} \). Applying Lemma 1, we have another result similar to the Corollary.

**Proposition.** A necessary and sufficient condition that \( U \) be induced by a continuous mapping of \( K \) into itself is that \( F(U) \) be total in \( C(K) \).

**Proof.** Since the set \( \{ f \in C(K); |f| = 1 \} \) is total in \( C(K) \), the necessity of the condition follows easily. Conversely suppose \( F(U) \) is total in \( C(K) \). Then, by Lemma 1, for any \( f \) and \( g \) in \( C(K) \) we have \( U(fg) = (Uf)(Ug) \). Hence Theorem 10.6 of [1] completes the proof.

**Bibliography**


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