PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

POSITIVE CONTRACTIONS ON L_1-SPACES

RYOTARO SATO

ABSTRACT. Let T be a positive linear operator on L_1 of a probability space such that $\|T\|_1 \leqq 1$. In this note we consider the following question: Under what condition is T multiplicative on L_∞?

Let $(X, \mathscr{F}, m)$ be a probability space and $L_1 = L_1(X, \mathscr{F}, m)$ the Banach space of equivalence classes of integrable complex-valued functions on X. Let T be a positive linear operator on L_1 such that $\|T\|_1 \leqq 1$, and define $G(T) = \{f \in L_1; |f| = |Tf| = 1\}$. The main purpose of this note is to prove the following:

THEOREM. *If $G(T)$ is total in L_1, i.e., the linear manifold generated by $G(T)$ is dense in L_1 in the norm topology, then for any bounded functions f and g we have $T(fg) = (Tf)(Tg)$.*

As a corollary of the Theorem we have the following result; essentially the same idea has been used by Halmos [2, p. 45] to find a necessary and sufficient condition for a unitary operator on $L_2 = L_2(X, \mathscr{F}, m)$ to be induced by an automorphism of the measure algebra $(\mathscr{F}(m), m)$ associated with $(X, \mathscr{F}, m)$.

COROLLARY. *A necessary and sufficient condition that T be induced by a homomorphism of the measure algebra $(\mathscr{F}(m), m)$ into itself is that $G(T)$ be total in L_1.*

For the proof of the theorem we need some lemmas.

LEMMA 1. *Let K be a compact Hausdorff space and $C(K)$ the Banach algebra of all complex-valued continuous functions f on K with norm $\|f\|_u = \sup\{|f(s)|; s \in K\}$. Let U be a positive linear operator on $C(K)$ such that $U1 = 1$. Let $f, g \in C(K)$, $|f| = |g| = 1$, and $|Uf| = |Ug| = 1$. Then $U(fg) = (Uf)(Ug)$.*

Received by the editors February 28, 1973 and, in revised form, May 10, 1973.

AMS (MOS) subject classifications (1970). Primary 47B55, 47A35; Secondary 28A65, 46J10.

Key words and phrases. Positive contraction, multiplicative, measure algebra, homomorphism of a measure algebra.

PROOF. For a fixed point s in K, the linear functional $h \to (Uh)(s)$ is positive and continuous. The Riesz representation theorem implies that there exists a positive Radon measure $m(s, dt)$ on K such that

$$(Uh)(s) = \int h(t)m(s, dt), \qquad h \in C(K).$$

Since $(Uf)(s) = \int f(t)m(s, dt)$ and $|f| = |Uf| = 1$, it follows that $f = (Uf)(s)$ almost everywhere with respect to the measure $m(s, dt)$. Similarly $g = (Ug)(s)$ almost everywhere with respect to $m(s, dt)$. Consequently $U(fg)(s) = \int f(t)g(t)m(s, dt) = (Uf)(s)(Ug)(s)$, and the lemma is proved.

Applying the lemma above, we have the following

LEMMA 2. *Let $|f| = |g| = 1$ and $|Tf| = |Tg| = 1$. Then $T(fg) = (Tf)(Tg)$.*

PROOF. Since $1 = |Tf| \leqq T|f| = T1$ and $\|T\|_1 \leqq 1$, we observe $T1 = 1$. Thus T maps $L_\infty = L_\infty(X, \mathscr{F}, m)$ into itself and $\|T\|_\infty \leqq 1$. The Gel'fand-Naimark theorem implies that L_∞ is isometrically isomorphic with $C(K)$ for some compact Hausdorff space K. Hence we may associate with T a linear operator U on $C(K)$ with $U1 = 1$. It is now easy to check that U is positive and that if $\hat{f}$ and $\hat{g}$ are the functions in $C(K)$ corresponding to f and g, respectively, then $|\hat{f}| = |\hat{g}| = 1$ and $|U\hat{f}| = |U\hat{g}| = 1$. Therefore $U(\hat{f}\hat{g}) = (U\hat{f})(U\hat{g})$ by Lemma 1. This completes the proof of Lemma 2.

PROOF OF THEOREM. Let M denote the linear manifold generated by $G(T)$, and let f, g be bounded functions. Since M is a dense subset of L_1 consisting of bounded functions, there exist $f_n, g_n \in M$, $n = 1, 2, \cdots$, such that

$$\lim_n \|f_n g_n - fg\|_1 = \lim_n \|f_n - f\|_1 = \lim_n \|g_n - g\|_1 = 0.$$

Lemma 2 implies that $T(f_n g_n) = (Tf_n)(Tg_n)$ for each n, and hence we have $T(fg) = (Tf)(Tg)$. This completes the proof.

PROOF OF COROLLARY. Since the necessity of the condition is obvious, we prove here only the sufficiency of it.

Suppose $G(T)$ is total in L_1. The theorem implies that if $A \in \mathscr{F}$ then $(T1_A)^2 = T1_A$. This shows that $T1_A = 1_B$ for some $B \in \mathscr{F}$. Thus if we set $\varphi(A) = B$, then φ defines a mapping of the measure algebra $(\mathscr{F}(m), m)$ into itself. Since $\|T\|_1 \leqq 1$ and $T1 = 1$, it follows that

$$\varphi(A - B) = \varphi(A) - \varphi(B), \qquad \varphi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \varphi(A_n)$$

and

$$m(A) = m(\varphi(A)),$$

whenever A, B, and A_n, $n = 1, 2, \cdots$, are elements of $(\mathscr{F}(m), m)$. That is to say, φ is a homomorphism of $(\mathscr{F}(m), m)$ into itself, and the Corollary is proved.

Let K, $C(K)$, and U be the same as in Lemma 1, and define $F(U)=\{f\in C(K); |f|=|Uf|=1\}$. Applying Lemma 1, we have another result similar to the Corollary.

PROPOSITION. *A necessary and sufficient condition that U be induced by a continuous mapping of K into itself is that $F(U)$ be total in $C(K)$.*

PROOF. Since the set $\{f\in C(K); |f|=1\}$ is total in $C(K)$, the necessity of the condition follows easily. Conversely suppose $F(U)$ is total in $C(K)$. Then, by Lemma 1, for any f and g in $C(K)$ we have $U(fg)=(Uf)(Ug)$. Hence Theorem 10.6 of [1] completes the proof.

BIBLIOGRAPHY

1. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

2. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc. Japan, no. 3, Math. Soc. of Japan, Tokyo, 1956. MR **20** #3958.

DEPARTMENT OF MATHEMATICS, JOSAI UNIVERSITY, SAKADO, SAITAMA 350-02, JAPAN