

POSITIVE CONTRACTIONS ON L_1 -SPACES

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ABSTRACT. Let T be a positive linear operator on L_1 of a probability space such that $\|T\|_1 \leq 1$. In this note we consider the following question: Under what condition is T multiplicative on L_∞ ?

Let (X, \mathcal{F}, m) be a probability space and $L_1 = L_1(X, \mathcal{F}, m)$ the Banach space of equivalence classes of integrable complex-valued functions on X . Let T be a positive linear operator on L_1 such that $\|T\|_1 \leq 1$, and define $G(T) = \{f \in L_1; |f| = |Tf| = 1\}$. The main purpose of this note is to prove the following:

THEOREM. *If $G(T)$ is total in L_1 , i.e., the linear manifold generated by $G(T)$ is dense in L_1 in the norm topology, then for any bounded functions f and g we have $T(fg) = (Tf)(Tg)$.*

As a corollary of the Theorem we have the following result; essentially the same idea has been used by Halmos [2, p. 45] to find a necessary and sufficient condition for a unitary operator on $L_2 = L_2(X, \mathcal{F}, m)$ to be induced by an automorphism of the measure algebra $(\mathcal{F}(m), m)$ associated with (X, \mathcal{F}, m) .

COROLLARY. *A necessary and sufficient condition that T be induced by a homomorphism of the measure algebra $(\mathcal{F}(m), m)$ into itself is that $G(T)$ be total in L_1 .*

For the proof of the theorem we need some lemmas.

LEMMA 1. *Let K be a compact Hausdorff space and $C(K)$ the Banach algebra of all complex-valued continuous functions f on K with norm $\|f\|_\infty = \sup\{|f(s)|; s \in K\}$. Let U be a positive linear operator on $C(K)$ such that $U1 = 1$. Let $f, g \in C(K)$, $|f| = |g| = 1$, and $|Uf| = |Ug| = 1$. Then $U(fg) = (Uf)(Ug)$.*

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PROOF. For a fixed point s in K , the linear functional $h \rightarrow (Uh)(s)$ is positive and continuous. The Riesz representation theorem implies that there exists a positive Radon measure $m(s, dt)$ on K such that

$$(Uh)(s) = \int h(t)m(s, dt), \quad h \in C(K).$$

Since $(Uf)(s) = \int f(t)m(s, dt)$ and $|f| = |Uf| = 1$, it follows that $f = (Uf)(s)$ almost everywhere with respect to the measure $m(s, dt)$. Similarly $g = (Ug)(s)$ almost everywhere with respect to $m(s, dt)$. Consequently $U(fg)(s) = \int f(t)g(t)m(s, dt) = (Uf)(s)(Ug)(s)$, and the lemma is proved.

Applying the lemma above, we have the following

LEMMA 2. *Let $|f| = |g| = 1$ and $|Tf| = |Tg| = 1$. Then $T(fg) = (Tf)(Tg)$.*

PROOF. Since $1 = |Tf| \leq T|f| = T1$ and $\|T\|_1 \leq 1$, we observe $T1 = 1$. Thus T maps $L_\infty = L_\infty(X, \mathcal{F}, m)$ into itself and $\|T\|_\infty \leq 1$. The Gel'fand-Naimark theorem implies that L_∞ is isometrically isomorphic with $C(K)$ for some compact Hausdorff space K . Hence we may associate with T a linear operator U on $C(K)$ with $U1 = 1$. It is now easy to check that U is positive and that if \hat{f} and \hat{g} are the functions in $C(K)$ corresponding to f and g , respectively, then $|\hat{f}| = |\hat{g}| = 1$ and $|U\hat{f}| = |U\hat{g}| = 1$. Therefore $U(\hat{f}\hat{g}) = (U\hat{f})(U\hat{g})$ by Lemma 1. This completes the proof of Lemma 2.

PROOF OF THEOREM. Let M denote the linear manifold generated by $G(T)$, and let f, g be bounded functions. Since M is a dense subset of L_1 consisting of bounded functions, there exist $f_n, g_n \in M, n = 1, 2, \dots$, such that

$$\lim_n \|f_n g_n - fg\|_1 = \lim_n \|f_n - f\|_1 = \lim_n \|g_n - g\|_1 = 0.$$

Lemma 2 implies that $T(f_n g_n) = (Tf_n)(Tg_n)$ for each n , and hence we have $T(fg) = (Tf)(Tg)$. This completes the proof.

PROOF OF COROLLARY. Since the necessity of the condition is obvious, we prove here only the sufficiency of it.

Suppose $G(T)$ is total in L_1 . The theorem implies that if $A \in \mathcal{F}$ then $(T1_A)^2 = T1_A$. This shows that $T1_A = 1_B$ for some $B \in \mathcal{F}$. Thus if we set $\varphi(A) = B$, then φ defines a mapping of the measure algebra $(\mathcal{F}(m), m)$ into itself. Since $\|T\|_1 \leq 1$ and $T1 = 1$, it follows that

$$\varphi(A - B) = \varphi(A) - \varphi(B), \quad \varphi\left(\bigcup_{n=1}^\infty A_n\right) = \bigcup_{n=1}^\infty \varphi(A_n)$$

and

$$m(A) = m(\varphi(A)),$$

whenever A, B , and $A_n, n = 1, 2, \dots$, are elements of $(\mathcal{F}(m), m)$. That is

to say, φ is a homomorphism of $(\mathcal{F}(m), m)$ into itself, and the Corollary is proved.

Let K , $C(K)$, and U be the same as in Lemma 1, and define $F(U) = \{f \in C(K); |f| = |Uf| = 1\}$. Applying Lemma 1, we have another result similar to the Corollary.

PROPOSITION. *A necessary and sufficient condition that U be induced by a continuous mapping of K into itself is that $F(U)$ be total in $C(K)$.*

PROOF. Since the set $\{f \in C(K); |f| = 1\}$ is total in $C(K)$, the necessity of the condition follows easily. Conversely suppose $F(U)$ is total in $C(K)$. Then, by Lemma 1, for any f and g in $C(K)$ we have $U(fg) = (Uf)(Ug)$. Hence Theorem 10.6 of [1] completes the proof.

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