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ON CLOSED SETS OF ORDINALS

HARVEY FRIEDMAN[1]

ABSTRACT. We prove that every stationary set of countable ordinals contains arbitrarily long countable closed subsets.

Call a set A of ordinals *closed* if and only if every nonempty subset of A which has an upper bound in A has its least upper bound in A. It is well known that there are $B \subset \omega_1$ such that neither B nor $\omega_1 - B$ contains an uncountable closed subset. A consequence of what we prove here is that for every $B \subset \omega_1$, either B or $\omega_1 - B$ contains arbitrarily long countable closed subsets.

Call a set A of ordinals *κ-stationary* if and only if $A \subset \kappa$ and A intersects every closed subset of κ of power κ. We can restate the above well-known theorem as follows: There is an A such that A and $\omega_1 - A$ are both ω_1-stationary.[2]

We will prove here that every ω_1-stationary set contains arbitrarily long, countable, closed subsets.

Is there a cardinal κ such that for all $A \subset \kappa$, either A or $\kappa - A$ contains an uncountable closed subset? Is this true for $\kappa = \omega_2$? Karel Prikry and the author noticed that, in any case, the statement for $\kappa = \omega_2$ cannot be proved true in ZFC.[3]

THEOREM. *Every ω_1-stationary set contains arbitrarily long countable closed subsets.*

PROOF. Let A be ω_1-stationary. We prove by induction on $\alpha < \omega_1$ that A has a closed subset of length α. Let the induction hypothesis be that

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[2] In fact, Solovay [1] proves that for uncountable regular cardinals κ, every κ-stationary set is the union of κ disjoint κ-stationary sets.

[3] By adding an $f:\omega_2 \to \{0, 1\}$ generic with respect to the partial ordering of countable partial $g:\omega_2 \to \{0, 1\}$. If the ground model satisfies ZFC$+2^\omega = \omega_1$, then in the forcing extension cardinals are preserved, $\{\alpha : f(\alpha) = 1\}$ and $\{\alpha : f(\alpha) = 0\}$ contain no uncountable closed subsets, and $2^\omega = \omega_1$ holds.

for all $\beta<\alpha$ and for each $\gamma<\omega_1$, there is a closed subset $B\subset A$ of length β, all of whose elements are $>\gamma$.

Case 1. α is a limit ordinal $<\omega_1$. Choose $\beta_0<\beta_1<\cdots<\alpha$, with $\sup_n(\beta_n)=\alpha$. Let $\gamma<\omega_1$. By the induction hypothesis, let $B_0\subset A$, B_0 of length β_0+1, B_0 closed, $(\forall\beta\in B_0)$ $(\beta>\gamma)$. Let $B_{n+1}\subset A$, B_{n+1} of length $\beta_{n+1}+1$, B_{n+1} closed, $(\forall\beta\in B_{n+1})$ $(\beta>\sup(B_n))$. Then set $B=\bigcup_n B_n$. B has the desired properties.

Case 2. $\alpha=\delta+2$, $\alpha<\omega_1$. Let $\gamma<\omega_1$. By the induction hypothesis, let $B_0\subset A$ be closed, of length $\delta+1$, and $(\forall\beta\in B_0)$ $(\beta>\gamma)$. Let $\lambda\in A$ with $\lambda>\sup(B_0)$. Put $B=B_0\cup\{\lambda\}$. B has the desired properties.

Case 3. $\alpha=\lambda+1$, λ a limit ordinal $<\omega_1$. Let $\lambda_0<\lambda_1<\cdots<\lambda$, $\sup_n(\lambda_n)=\lambda$. Let $\gamma<\omega_1$. By the induction hypothesis, define a sequence of sets B_ξ, $\xi<\omega_1$, such that

(a) $(\forall\beta\in B_0)$ $(\beta>\gamma)$

(b) if $\xi_1<\xi_2$ then $(\forall\beta_1\in B_{\xi_1})(\forall\beta_2\in B_{\xi_2})(\beta_1<\beta_2)$

(c) each B_ξ is a closed subset of A of length λ.

Define $f:\omega_1\to\omega_1$ by $f(\xi)=\sup(\bigcup_{\sigma<\xi}B_\sigma)$. Note that the range of f on limit ordinals is an uncountable closed set. Since A is stationary, let τ be a countable limit ordinal with $f(\tau)\in A$. Choose $\tau_0<\tau_1<\cdots<\tau$ with $\sup_n(\tau_n)=\tau$. Let C_n be the first λ_n+1 elements of B_{τ_n}. Then set $B^*=\bigcup_n C_n$. B^* is a closed subset of A of length λ, and $\sup(B^*)=f(\tau)\in A$. Hence $B=B^*\cup\{f(\tau)\}$ is a closed subset of length at least α, all of whose elements are $>\gamma$, and we are done.

The referee has kindly forwarded the following remarks concerning the problems raised on the first page of this paper.

Let us say that a cardinal $K>\omega$ has the property F (briefly, $F(K)$) if for every subset A of K either A or $K-A$ contains a closed subset of order type ω_1.

(1) Silver has shown that the Jensen principle $\Box_{\omega_1}$ implies $\neg F(\omega_2)$. Since $\neg\Box_{\omega_1}\to$"ω_2 is Mahlo in L" this gives a lower bound on the proof-theoretic strength of ZFC$+F(\omega_2)$.

(2) Silver has also observed that in any Cohen extension of any model M of ZFC obtained by generically collapsing ω_1^M to ω, $F(K)$ fails for all uncountable M-cardinals K. (For A take $\{\alpha:cf^M(\alpha)=\omega$ and $\alpha<K\}$.)

(3) Solovay has generalized Silver's proof in (1) above to show that, in L, $F(K)$ fails for all cardinals $K>\omega$.

Karel Prikry has informed the author that he has independently shown that $F(K)$ fails for all cardinals $K>\omega$, in L.

Reference

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Department of Mathematics, Stanford University, Stanford, California 94305

Department of Mathematics, State University of New York at Buffalo, Amherst, New York 14226