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REGULARITY IN TERMS OF REDUCTIONS IN LOCAL NOETHER LATTICES

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ABSTRACT. An (n, d)-sequence in a local Noether lattice (L, M) is a sequence of words which satisfy certain factorization properties. If (L, M) satisfies the union condition, there exist (n, d)-sequences which can be extended to minimal bases for the powers of M. Consequently, if (L, M) satisfies the union condition, (L, M) is regular if and only if M is a minimal reduction.

Some time ago, Northcott and Rees [**8**, pp. 147–152] presented a result characterizing certain regular local Noetherian rings in terms of reductions of the maximal ideal. An ideal A is a reduction of an ideal B if $A \subseteq B$ and $AB^n = B^{n+1}$ for some integer $n > 0$. This ideal-theoretic definition is easily generalized to Noether lattices. However, the proof of the result in [**8**] does not generalize since it involves the use of polynomial rings for which, to our knowledge, we have no suitable analog in the theory of Noether lattices.

In this note, we overcome this obstacle by introducing (n, d)-sequences, sequences of words in the underlying multiplicative semigroup of a Noether lattice. We show that a local Noether lattice satisfying the union condition is regular if and only if its maximal element is a minimal reduction.

1. **(n, d)-sequences.** Throughout this section $S^* = \{x_1, x_2, \cdots, x_m, \cdots\}$ is a countably infinite set and S is the free monoid on S^*.

DEFINITION 1.1. If $d \geqq 2$ and $U, V_1, \cdots, V_t$ are words in S, we say that the sequence $V_1, \cdots, V_t$ is *d-factorable from U* if each V_i has at least one of a set of fewer than d factors distinct from the factors of U. $W_1, W_2, \cdots, W_q$ is an *(n, d)-sequence* if the W_i are all of degree n and $W_{l+1}, \cdots, W_q$ is d-factorable from W_l for $1 \leqq l < q$.

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The degree of a word $\prod_j x_j^{a(j)}$ is $\sum_j a(j)$. If x^a appears in a word with $a>0$ then x is a factor of that word. We use $C(m, n)$ to denote the binomial coefficient and $|A|$ to denote the cardinality of the set A.

When $\Gamma: W_1, \cdots, W_q$ is an (n, d)-sequence consisting of products of principal elements $\{E_i\}$ in a local Noether lattice L, the d-factorability of Γ insures that for $1 \leqq l < q$, $\bigvee_{i=l+1}^q W_i \leqq T(\bigvee_{i=l+1}^q W_i')$ where T is the join of fewer than d of the E_i distinct from the factors of W_l. This factorization property guarantees in our context that $\bigvee W_i$ can be extended to a minimal base for A^n where $A = \bigvee E_i$ (Lemma 2.5). Moreover, when A is the maximal element of L, the existence of (n, d)-sequences of length $C(n+d-1, d-1)$ forces minimal bases for A^n to be long enough to make L regular (Theorem 2.9). We now demonstrate the existence of such (n, d)-sequences.

REMARK 1.2. If $\Gamma: W_1, \cdots, W_q$ is an (n, d)-sequence with factors from $F \subseteq S^*$ and $t = x_k$ is any element in $S^* \backslash F$, then $t^a\Gamma: t^a W_1, \cdots, t^a W_q$ is an $(n+a, d)$-sequence for any $a \geqq 0$. If $F = \{x_1, \cdots, x_m\}$ and ϕ is any injection from $\{1, 2, \cdots, m\}$ into the nonzero positive integers, then the sequence, denoted Γ_ϕ, obtained by substituting $\phi(i)$ for i, $1 \leqq i \leqq m$, as subscripts in Γ is an (n, d)-sequence with factors from $F_\phi = \{x_{\phi(1)}, \cdots, x_{\phi(m)}\}$. Note that we may assume that (n, d)-sequences have factors from $\{x_1, \cdots, x_p\}$ for some p.

THEOREM 1.3. *For all $n \geqq 1$ and for all $d \geqq 2$, there exists an (n, d)-sequence of words from S of length $C(n+d-1, d-1)$.*

PROOF. We proceed by induction on n and d. For $n \geqq 1$, the sequence

$$x_1^n, x_n^n, x_{n-1}^{n-1}x_n, x_{n-2}^{n-2}x_{n-1}x_n, \cdots, x_2^2 x_3 \cdots x_n, x_1 x_2 \cdots x_n$$

is an $(n, 2)$-sequence of length $C(n+1, 1)$. For $d \geqq 2$, the sequence $x_1, x_2, \cdots, x_d$ is a $(1, d)$-sequence of length $C(d, d-1)$.

By our induction hypothesis, for $d \geqq 3$, let $\Gamma: W_1, W_2, \cdots, W_q$ be an (n, d)-sequence with factors from $F = \{x_1, \cdots, x_m\}$ of length $q = C(n+d-1, d-1)$ and $\Delta: V_1, V_2, \cdots, V_r$ be an $(n+1, d-1)$-sequence of length $r = C((n+1)+(d-2), d-2)$ with factors from $G = \{x_1, \cdots, x_p\}$. If $\phi(i) = p+i$, $1 \leqq i \leqq m$, and $t = x_{m+p+1}$, by Remark 1.2, $t\Gamma_\phi: tW_1', tW_2', \cdots, tW_q'$ is an $(n+1, d)$-sequence which we write as $V_{r+1}, \cdots, V_{r+q}$.

We claim that the sequence $V_1, V_2, \cdots, V_{r+q}$ is an $(n+1, d)$-sequence. Clearly, the degree of each V_i is $n+1$. If $r+1 \leqq l < r+q$, then since $t\Gamma_\phi$ is an $(n+1, d)$-sequence, $V_{l+1}, \cdots, V_{r+q}$ is d-factorable from V_l. Moreover, t is a factor of $V_{r+1}, \cdots, V_{r+q}$ but not a factor of V_r; so, $V_{r+1}, \cdots, V_{r+q}$ is d-factorable from V_r. Finally, if $1 \leqq l < r$, since Δ is an $(n+1, d-1)$-sequence, there is a subset A of F such that $|A| \leqq d-2$, each of $V_{l+1}, \cdots, V_r$ has some element in A as a factor, and no element in A is a factor of V_l.

But then each of $V_{l+1}, \cdots, V_{r+q}$ has some element of $A \cup \{t\}$ as a factor and no element of $A \cup \{t\}$ is a factor of V_l. Hence, in this case $V_{l+1}, \cdots, V_{r+q}$ is d-factorable from V_l. Hence, $V_1, \cdots, V_{r+q}$ is an $(n+1, d)$-sequence of length

$$\begin{aligned} r + q &= C((n+1) + (d-2), d-2) + C(n+d-1, d-1) \\ &= C((n+1) + (d-1), d-1). \qquad \text{Q.E.D.} \end{aligned}$$

Let L be a local Noether lattice and $T^* = \{E_i\}$ be a countably infinite set of elements in L. Let T be the free commutative monoid on T^*. By Theorem 1.3, for all $n \geqq 1$ and $d \geqq 2$, there exists an (n, d)-sequence of length $C(n+d-1, d-1)$ with factors from T^*. Reading the juxtaposition multiplication on T as the multiplication in L, we obtain

COROLLARY 1.4. *Let L be a local Noether lattice and $T^* = \{E_i\}$ be a countably infinite subset of L. Then there exists an (n, d)-sequence of elements in L of length $C(n+d-1, d-1)$ for all $n \geqq 1$ and for all $d \geqq 2$.*

Again, using the technique of reading juxtaposition multiplication as multiplication of the semigroup; and conversely, we have

COROLLARY 1.5. *Let L, $\{E_i\}$, and S, $\{x_i\}$ be as above. There exists an (n, d)-sequence of length t in L with factors from $\{E_1, \cdots, E_p\}$ if and only if there exists an (n, d)-sequence of length t in S with factors from $\{x_1, \cdots, x_p\}$.*

2. **Reductions and regularity.** We now turn to our application of (n, d)-sequences to the study of reductions in certain local Noether lattices. Throughout this section (L, M) is a local Noether lattice with maximal element M.

DEFINITION 2.1. An element B in L is a *reduction* of an element, A, in L $(B \rho A)$ if $B \leqq A$ and $BA^n = A^{n+1}$ for some $n > 0$. If B is a reduction of A, then B is a *minimal reduction* of A if $C \leqq B$ and $C \rho A$ implies that $C = B$.

The following theorem is proved as in [**8**, pp. 146–147] with the appropriate lattice-theoretic interpretations.

THEOREM 2.2. *For A, B, and C in L:*

(1) *$C \rho B$ and $B \rho A$ implies $C \rho A$.*

(2) *$C \rho A$ if and only if $(C \vee AM) \rho A$.*

(3) *$B \rho A$ implies there exists a minimal reduction D of A such that $D \leqq B$.*

(4) *If C is a minimal reduction of A and $C \leqq B \leqq A$, then every minimal base for C can be extended to a minimal base for B.*

DEFINITION 2.3. L satisfies the *union condition* if given any finite collection of elements, $A_1, A_2, \cdots, A_k$, in L and any A in L such that $A \nleq A_i$ for $i=1, 2, \cdots, k$, there exists a principal element B in L such that $B \leqq A$ and $B \nleq A_i$ for $i=1, 2, \cdots, k$ [**6**, 1.7].

If R is a local Noetherian ring (commutative with identity) with an infinite residue field, then the lattice of ideals of R satisfies the union condition.

If $B \neq I$ is an element of L, let $l(B/BM)$ denote the number of elements in a minimal base for B (which equals the length of the composition series from BM to B [**5**]). For the remainder of this paper, we assume L satisfies the union condition. The union condition forces each element B in L with $l(B/BM)>1$ to have an infinite number of minimal bases. More precisely

LEMMA 2.4. *If B is an element of L such that $l(B/BM)=d>1$, then there exists a countably infinite set of principal elements $\{E_i\}$ such that any d of them forms a minimal base for B.*

PROOF. Suppose that $B=E_1 \vee E_2 \vee \cdots \vee E_d$ where the E_i's form a minimal base and that $E_1, E_2, \cdots, E_{d+n}$ have been chosen, $n \geqq 0$. Let

$$\{P_\alpha\} = \left\{ \bigvee_{j \in J_\alpha} E_j \quad BM \;\middle|\; J_\alpha \subseteq \{1, \cdots, d+n\} \text{ and } |J_\alpha| = d-1 \right\}.$$

By the union condition, since $B \nleq P_\alpha$ for each α, and $B \nleq BM$, there exists a principal element, $E_{d+n+1} \leqq B$, in L such that $E_{d+n+1} \nleq P_\alpha$ for each α and $E_{d+n+1} \nleq BM$. Since $l(B/BM)=d$, $E_{d+n+1} \vee (\bigvee_{j \in J_\alpha} E_j)$ forms a minimal base for B for each α. By induction our proof is complete. Q.E.D.

LEMMA 2.5. *If B is a minimal reduction of A in L with $l(B/BM)=d>1$ and if $\{E_i\}$ is a countably infinite set of principal elements for B as in Lemma 2.4, then any (n, d)-sequence in the subsemigroup of L generated by the E_i can be extended to a minimal base for A^n.*

PROOF. First we show that if $T=F_{m1}^{n(1)} F_{m2}^{n(2)} \cdots F_{mt}^{n(t)}$ is any product of the E_i's such that $\sum_j n(j)=n$, then $T \nleq SB^{n-1} \vee A^n M$ where S is the join of $d-1$ of the E_i's distinct from the F_{mj}. If $T \leqq SB^{n-1} \vee A^n M$, then

$$B^n \vee A^n M = [(F_{m1} \vee S)^{n(1)} (F_{m2} \vee S)^{n(2)} \cdots (F_{mt} \vee S)^{n(t)}] \vee A^n M$$

since the join of any d of the E_i's is B. Then $B^n \vee A^n M \leqq (T \vee SB^{n-1}) \vee A^n M = SB^{n-1} \vee A^n M \leqq B^n \vee A^n M$. So, if $BA^r = A^{r+1}$,

$$\begin{aligned} A^{n+r} &= B^n A^r \leqq B^n A^r \vee A^{n+r} M = (B^n \vee A^n M) A^r \\ &= SB^{n-1} A^r \vee A^{n+r} M = SA^{n+r-1} \vee A^{n+r} M. \end{aligned}$$

Hence, $A^{n+r}=SA^{n+r-1}$ with S strictly less than B. This contradicts the fact that B is a minimal reduction of A.

Now let $W_1, \cdots, W_q$ be an (n, d)-sequence with factors from $\{E_1, \cdots, E_p\}$. Then each W_i is a principal element in L. If $W_q \leqq A^nM$, then

$$W_q \leqq (E_{p+1} \vee E_{p+2} \vee \cdots \vee E_{p+d-1})B^{n-1} \vee A^nM,$$

contradicting the first paragraph. So $W_q \nleqq A^nM$. Suppose that $W_t \leqq W_{t+1} \vee \cdots \vee W_q \vee A^nM$ where $1 \leqq t < q$. Since $W_{t+1}, \cdots, W_q$ is d-factorable from W_t, there exists a subset, $\{F_1, \cdots, F_k\}$, of $\{E_1, \cdots, E_p\}$ with $k \leqq d-1$ such that each of the elements $W_{t+1}, \cdots, W_q$ has one of the F_i as a factor but no F_i is a factor of W_t. Therefore, $W_t \leqq (F_1 \vee \cdots \vee F_k)B^{n-1} \vee A^nM$. Again this is a contradiction. Hence, $W_1, \cdots, W_q$ can be extended to a minimal base for A^n. Q.E.D.

DEFINITION 2.6. Principal elements, $E_1, \cdots, E_d$ in L are *analytically independent in A* if the join of all products of $E_1, \cdots, E_d$ of degree n can be extended to a minimal base for A^n for all $n \geqq 1$. If $E_1, \cdots, E_d$ are analytically independent in $E_1 \vee \cdots \vee E_d$, we say that $E_1, \cdots, E_d$ are *analytically independent*.

REMARK 2.7. Let R be a commutative local Noetherian ring with identity and maximal ideal $\mathfrak{M}$. Then $t_1, \cdots, t_r$ in $\mathfrak{M}$ are analytically independent in the sense of [7, p. 67] if and only if the ideals $(t_1), \cdots, (t_r)$ are analytically independent in $L(R)$, the lattice of ideals of R, in the sense of Definition 2.6.

PROOF. Suppose $(t_1), \cdots, (t_r)$ are not analytically independent in $L(R)$. Then some product $\prod (t_i)^{\xi(i)}$ of degree, say, n is contained in the join (ideal sum) of different products of the ideals (t_i) of degree n join $(t_1, \cdots, t_r)^n\mathfrak{M}$. But then $\prod t_i^{\xi(i)}$ is equal to $\varphi(t_1, \cdots, t_r)$ where φ is a form of degree n; and $\varphi(t_1, \cdots, t_r) - \prod t_i^{\xi(i)}$ gives a form of degree n which equals 0 when evaluated at $t_1, \cdots, t_r$ but all of the coefficients are not in $\mathfrak{M}$.

Conversely, if $(t_1), \cdots, (t_r)$ are analytically independent in $L(R)$ and $\Phi(z_1, \cdots, z_r)$ is a form of degree n, suppose $\Phi(t_1, \cdots, t_r)=0$ and the coefficient a of $\prod t_i^{\xi(i)}$ is not in $\mathfrak{M}$. Then $\prod (t_i)^{\xi(i)}$ is contained in the join of different products of the (t_i) of degree n since $(a)=R$ which is a contradiction. Q.E.D.

We agree that 0 in L is analytically independent in A for any A in L. Note that $E_1, \cdots, E_d$ are analytically independent in A if and only if $l((B^n \vee A^nM)/A^nM) = C(n+d-1, d-1)$ for all $n \geqq 1$ where $B = E_1 \vee \cdots \vee E_d$.

Let $Q_1, \cdots, Q_n$ be a system of parameters for L and let $Q = Q_1 \vee \cdots \vee Q_n$. Then there exists a polynomial $p_Q(x)$ with rational coefficients such that $p_Q(k) = l(Q^k/Q^kM)$ for all sufficiently large k. The degree of $p_Q(x)$, the Hilbert polynomial for Q, is $n-1$ [5, §3].

Let J_m be the set of products of the Q_i of degree m. Let us agree to write the elements of J_m as the ordered product, $Q_1^{t(1)} \cdots Q_n^{t(n)}$ where some of the exponents may be zero. (If A is in L, $A^0=I$.) For A in J_m and $S \subseteq J_m$, we say that A depends on S if $A \leqq (\bigvee S) \vee Q^m M$ where we set $\bigvee \varnothing = 0$. If we order the elements of J_m by the lexicographic order of the exponent n-tuples, $\bigvee J_m$ is a minimal base for Q^m if and only if for each A in J_m, A does not depend on the elements that precede it.

THEOREM 2.8. *If $Q_1, \cdots, Q_n$ is a system of parameters for (L, M), then $Q_1, \cdots, Q_n$ are analytically independent.*

PROOF. With notation as above, suppose that $m \geqq 1$ and A is an element in J_m such that A depends on the set $\mathfrak{I}$ of elements that precede it in the lexicographic n-tuple order. Then for any B in J_r, $BA \leqq B(\bigvee \mathfrak{I}) \vee BQ^m M \leqq (\bigvee B\mathfrak{I}) \vee Q^{m+r} M$ where $B\mathfrak{I} = \{BC \mid C \in \mathfrak{I}\}$. So, BA depends on the elements which precede it in J_{m+r}. Hence, if $q > m$ and B is in J_{q-m}, then BA can be deleted from $\bigvee J_q$ in forming a minimal base for Q^q from the elements of J_q. There are $C((q-m)+(n-1), n-1)$ such BA's which can be deleted from J_q. Hence,

$$\begin{aligned} l(Q^q/Q^q M) \leqq\ & C(q + (n-1), n-1) \\ & - C((q-m) + (n-1), n-1) \quad \text{for all } q > m. \end{aligned}$$

If $p(x)$ is the Hilbert polynomial for Q, this inequality implies that the degree of $p(x)$ is less than $n-1$. This contradiction proves the theorem. Q.E.D.

THEOREM 2.9. *Let (L, M) be a local Noether lattice satisfying the union condition. Then the following are equivalent:*

(1) *M is a minimal reduction of M.*

(2) *Any minimal base for M is analytically independent.*

(3) *(L, M) is a regular local Noether lattice.*

PROOF. (1) *implies* (2). If $M=0$, the result is clear. If $M \neq 0$ is a principal element, then since $M^n \not\leqq M^{n+1}$ for any $n \geqq 1$, M is analytically independent in M.

Suppose $M = M_1 \vee M_2 \vee \cdots \vee M_d$ where the M_i are principal elements forming a minimal base for M and $d \geqq 2$. Clearly, $l(M^n/M^{n+1}) \leqq C(n+d-1, d-1)$ for all $n \geqq 1$ since the products of degree n in $M_1, \cdots, M_d$ join to form M^n. Since L satisfies the union condition, by Lemma 2.4, there exists an infinite collection of principal elements, $\{M_i\}$, such that any d of them forms a minimal base for M. By Corollary 1.4, there exists an (n, d)-sequence Γ in L with factors from the M_i's of length $C(n+d-1, d-1)$. So, by Lemma 2.5, Γ can be extended to a minimal

base for M^n. Hence, $l(M^n/M^{n+1}) \geqq C(n+d-1, d-1)$; and $M_1, \cdots, M_d$ are analytically independent.

(2) *implies* (3). Let $M_1, \cdots, M_d$ be a minimal base for M. Since the M_i are analytically independent in M, $l(M^n/M^{n+1}) = C(n+d-1, d-1)$ for all $n \geqq 1$. Hence if $p(x)$ is the Hilbert polynomial for M, $p(n) = C(n+d-1, d-1)$. From [**5**, 3.9], the altitude of L is d. Therefore, (L, M) is a regular local Noether lattice.

(3) *implies* (1). From Theorem 2.8, every regular system of parameters for L is analytically independent. Hence, using the techniques in the proof of [**7**, §4, Theorem 1] we conclude that M is a minimal reduction of M. Q.E.D.

From Theorem 2.9, we obtain a remark on (n, d)-sequences.

COROLLARY 2.10. *If S^* and S are as in* §1, *then the maximum length of an (n, d)-sequence in S is $C(n+d-1, d-1)$.*

PROOF. Suppose there exists an (n, d)-sequence in S of length $t > C(n+d-1, d-1)$. Let (L, M) be a regular local Noether lattice of altitude d satisfying the union condition. Then by Theorem 2.9, M is a minimal reduction of M; and there exists an infinite sequence $\{M_i\}$ of principal elements such that any d of them is a regular system of parameters for L. By Corollary 1.5, there exists an (n, d)-sequence with factors in the M_i's of length t. This (n, d)-sequence can be extended to a minimal base for M^n. Consequently, $l(M^n/M^{n+1}) \geqq t > C(n+d-1, d-1) = l(M^n/M^{n+1})$. Q.E.D.

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