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EXTENSIONS OF THE INDEX IN FACTORS OF TYPE II_∞

MICHAEL GARTENBERG

ABSTRACT. In this paper we show that the analytic index has no continuous extension to those operators in a factor of type II_∞ on a separable Hilbert space which are not semi-Fredholm in the Breuer sense. A similar result has already been proved by Coburn and Lebow [3] for factors of type I_∞. Here we use Breuer's generalized Fredholm theory to extend their result to the more general setting.

1. **Definitions and preliminaries.** As usual, $\mathscr{B}(\mathscr{H})$ denotes the algebra of all bounded operators on the separable Hilbert space $\mathscr{H}$. A $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in the weak operator topology is called a von Neumann algebra. If the center of $\mathscr{A}$ consists precisely of scalar multiples of the identity, then $\mathscr{A}$ is called a factor. For E, F in $\mathscr{P}(\mathscr{A})$, the set of all projection operations in $\mathscr{A}$, we write $E \leqq F \Leftrightarrow EF = E$. The equivalence relation $\sim$ on $\mathscr{P}(\mathscr{A})$ is defined by $E \sim F$ if and only if there is a partially isometric operator U in $\mathscr{A}$ such that $E = U^*U$ and $F = UU^*$. Finally, an order relation $\precsim$ on $\mathscr{P}(\mathscr{A})$ is given by $E \precsim F$ if and only if there is an F' in $\mathscr{P}(\mathscr{A})$ such that $E \sim F' \leqq F$.

A projection operator E is said to be finite if it is not equivalent to any $F \in \mathscr{P}(\mathscr{A})$ where $F \leqq E$ and $F \neq E$. Otherwise, E is said to be infinite. If the identity of a von Neumann algebra $\mathscr{A}$ is a finite (infinite) projection, then $\mathscr{A}$ is called finite (infinite).

We follow Breuer's generalization of the concepts of compact and Fredholm operators to a von Neumann algebra $\mathscr{A}$. For $B \in \mathscr{A}$

$$N_B = \sup\{E \in \mathscr{P}(\mathscr{A}) : BE = 0\}$$

and

$$R_B = \inf\{E \in \mathscr{P}(\mathscr{A}) : EB = B\}$$

are called the null projection and range projection of B, respectively. We call B finite if R_B is a finite projection. If $\mathscr{K}$ is the norm closure of the

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set of finite elements of $\mathscr{A}$, then $\mathscr{K}$ is a closed two-sided ideal in $\mathscr{A}$. The elements of $\mathscr{K}$ are said to be compact relative to $\mathscr{A}$.

DEFINITION. Let $\mathscr{A}$ be a factor. An operator $B \in \mathscr{A}$ is said to be Fredholm relative to $\mathscr{A}$ if

(i) N_B is a finite projection, and

(ii) there is a finite projection E in $\mathscr{A}$ such that the range of $I-E$ is contained in the range of B.

Let $\mathscr{F}(\mathscr{A})$ be the set of Fredholm elements relative to a factor $\mathscr{A}$, and let Dim denote some fixed relative dimension function on $\mathscr{A}$ [**6**]. It follows from the definitions that if B is Fredholm relative to $\mathscr{A}$ then $\operatorname{Dim} N_B$ and $\operatorname{Dim} N_{B^*}$ are both finite. We can thus define the index, i, of a Fredholm element B relative to $\mathscr{A}$ by

$$i(B) = \operatorname{Dim} N_B - \operatorname{Dim} N_{B^*}.$$

The classical Fredholm theory is generalized by Breuer to compact and Fredholm operators relative to a factor. Specifically, if $\mathscr{A}$ is an infinite factor, then $B \in \mathscr{A}$ is Fredholm relative to $\mathscr{A}$ if and only if $\pi(B)$ is invertible in $\mathscr{A}/\mathscr{K}$; for $A, B \in \mathscr{F}(\mathscr{A})$, $i(A)=i(B)$ if and only if A and B lie in the same connected component of $\mathscr{F}(\mathscr{A})$ [**2**].

Let $\mathscr{G}(\mathscr{A}/\mathscr{K})$, $\mathscr{L}(\mathscr{A}/\mathscr{K})$ and $\mathscr{R}(\mathscr{A}/\mathscr{K})$ denote the open semigroups of invertible, left (but not right), and right (but not left) invertible elements of $\mathscr{A}/\mathscr{K}$, respectively. The elements of

$$\mathscr{S} = \pi^{-1}(\mathscr{G}(\mathscr{A}/\mathscr{K})) \cup \pi^{-1}(\mathscr{R}(\mathscr{A}/\mathscr{K})) \cup \pi^{-1}(\mathscr{L}(\mathscr{A}/\mathscr{K}))$$

are called semi-Fredholm relative to $\mathscr{A}$, and the notation $H(\mathscr{S})$ and $H(\pi(\mathscr{S}))$ is used to denote the set of connected components of $\mathscr{S}$ and $\pi(\mathscr{S})$, respectively.

We shall use the following in the proof of our main result.

PROPOSITION 1. *The map $\pi: H(\mathscr{S}) \to H(\pi(\mathscr{S}))$ defined by $\pi: \mathscr{C} \to \pi(\mathscr{C})$ for $\mathscr{C}$ in $H(\mathscr{S})$ is an isomorphism.*

PROOF. [**3**].

Essential in what follows is the notion of a cross section. Suppose f is a continuous mapping of X onto Y, where X and Y are Banach spaces. A continuous map $s: Y \to X$ is a continuous cross section of f if $f(s(y))=y$ for all $y \in Y$. A consequence of the theorem of Bartle and Graves [**1**] is that if π is the projection of $\mathscr{A}$ onto $\mathscr{A}/\mathscr{K}$, then π has a continuous cross section. An immediate result of this is the following.

THEOREM 1. *Let $\mathscr{C}$ be a component of $H(\mathscr{G}(\mathscr{A}/\mathscr{K}))$. Then* $\mathrm{Cl}(\pi^{-1}(\mathscr{C}))= \pi^{-1}(\overline{\mathscr{C}})$.

PROOF. Since $\pi^{-1}(\overline{\mathscr{C}})$ is a closed set containing $\pi^{-1}(\mathscr{C})$, $\mathrm{Cl}(\pi^{-1}(\mathscr{C}))\subset \pi^{-1}(\overline{\mathscr{C}})$. On the other hand, if $x\in\pi^{-1}(\overline{\mathscr{C}})$, then $\pi(x)\in\overline{\mathscr{C}}$. Suppose $\{y_n\}$ in $\mathscr{C}$ is such that $y_n\to\pi(x)$. If s is a continuous cross section of π, then $s(y_n)\to s(\pi(x))=x+k$ for some $k\in\mathscr{K}$. But then, $s(y_n)-k\to x$, and $s(y_n)=k\in\pi^{-1}(\mathscr{C})$. Hence, $x\in\mathrm{Cl}(\pi^{-1}(\mathscr{C}))$, and $\pi^{-1}(\overline{\mathscr{C}})=\mathrm{Cl}(\pi^{-1}(\mathscr{C}))$.

DEFINITION. A factor $\mathscr{A}$ is said to be of type II_∞ if the range of Dim on $\mathscr{P}(\mathscr{A})$ is the interval $[0,\infty]$.

2. **Main results.** In the following, we denote by $r(A)$ the closure of the range of the operator A. We use the same notation to indicate a closed subspace of $\mathscr{H}$ and the orthogonal projection on this subspace.

We need the following result of Feldman and Kadison:

PROPOSITION 2. *If $\mathscr{A}$ is a factor on a separable Hilbert space, then $\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$ consists of those $A\in\mathscr{A}$ such that for every $\varepsilon>0$ there is a projection $E\in\mathscr{A}$ containing the null projection of A with $\|AE\|<\varepsilon$ and $E\sim\mathscr{H}\ominus r(A(I-E))$.*

PROOF. [5].

Feldman and Kadison proceed to prove that if $\mathscr{A}=\mathscr{B}(\mathscr{H})$ then $A\notin\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$ if and only if A is the product of a regular operator and a partially isometric operator between subspaces of unequal codimension. For factors of type II_∞, we get the following modified result.

THEOREM 2. *If $\mathscr{A}$ is a factor of type II_∞ on a separable Hilbert space, and if $A\in\mathscr{A}$ is such that $A\notin\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$, then A is the product of a Fredholm operator relative to $\mathscr{A}$ of index zero and a partial isometry between subspaces of unequal relative codimension.*

PROOF. Suppose that $A\notin\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$, and let N be the null projection of A. We have that $\mathrm{Dim}\,N\neq\mathrm{Dim}\,\mathscr{H}\ominus r(A)$, for otherwise, $N\sim\mathscr{H}\ominus r(A)$, and choosing N as the E of Proposition 2, $A\in\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$. We can assume that $\mathrm{Dim}\,N<\mathrm{Dim}\,\mathscr{H}\ominus r(A)$, for, if not, we can deal with A^* since $N=\mathscr{H}\ominus r(A^*)$, and $N_{A^*}=\mathscr{H}\ominus r(A)$. Hence, $\mathrm{Dim}\,N<\infty$, and N is a finite projection.

If $A=U(A^*A)^{1/2}$ is the polar factorization of A [4], then U is a partial isometry mapping $r[(A^*A)^{1/2}]=r(A^*A)=\mathscr{H}\ominus N$ onto $r(A)$. Thus, the relative codimensions of the initial and final spaces of U are unequal. It remains to be shown that $T=(A^*A)^{1/2}$ is a Fredholm operator relative to $\mathscr{A}$ of index zero.

Since $N=N_T$ has finite relative dimension, it suffices to show that there exists a finite projection E such that the range of $I-E$ is contained in the range of T. Since $A\notin\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$, there exists, by Proposition 2, a constant $k>0$ such that if E is a projection in $\mathscr{A}$ with $\|AE\|\leqq k$, then E is not equivalent to $\mathscr{H}\ominus r[A(I-E)]$. If E is the spectral projection for T corresponding to the interval $[0, k]$, then $E\in\mathscr{A}$ [4], and $\|AE\|=\|UTE\|\leqq\|U\|\,\|TE\|\leqq k$. Since the subspace E contains N_T [2], E is not equivalent to $\mathscr{H}\ominus r[A(I-E)]$. We claim that E is a finite projection. Assume that E is infinite. Since $T(I-E)\geqq k(I-E)$, the operator T maps the space $I-E$ onto itself. Furthermore, we have that

$$r[U(I-E)] = r[UT(I-E)] = r[A(I-E)].$$

But if E is infinite and N finite, the space $r[U(E-N)]$ is infinite. Since N is contained in E, $I-E$ is orthogonal to $E-N$, and because U is an isometry on $\mathscr{H}\ominus N$, $r[U(E-N)]$ is also orthogonal to $r[U(I-E)]$. Hence, $\mathscr{H}\ominus r[U(I-E)]=\mathscr{H}\ominus r[A(I-E)]$ is infinite. Since $\mathscr{H}$ is separable, it follows that E is equivalent to $\mathscr{H}\ominus r[A(I-E)]$, which is impossible. Hence, E is finite. We also have that $r(I-E)\subset r(T)$ [2]. Therefore, T is Fredholm relative to $\mathscr{A}$, and since T is positive, its index is zero.

COROLLARY 1. *If $A\in\mathscr{A}$ satisfies the hypothesis of the theorem, then A is semi-Fredholm relative to $\mathscr{A}$ with $i(A)\neq 0$.*

PROOF. By the theorem, A is the product of a partial isometry U between subspaces of unequal relative codimension and a Fredholm operator relative to $\mathscr{A}$ of index zero. Since $\mathscr{H}$ is separable, one of these subspaces has finite relative codimension. Therefore, U is semi-Fredholm relative to $\mathscr{A}$ with $i(U)\neq 0$, and the same must be true for A.

Let $\mathscr{S}$ be the set of semi-Fredholm operators in a factor $\mathscr{A}$ of type II_∞ on a separable Hilbert space. Noting for $A\in\pi^{-1}(\mathscr{L}(\mathscr{A}/\mathscr{K}))$ (resp. $\pi^{-1}(\mathscr{R}(\mathscr{A}/\mathscr{K}))$) that $\operatorname{Dim} N_A<\infty$ (resp. $\operatorname{Dim} N_{A^*}<\infty$) [2], we write $\mathscr{F}_\infty=\pi^{-1}(\mathscr{L}(\mathscr{A}/\mathscr{K}))$ and $\mathscr{F}_{-\infty}=\pi^{-1}(\mathscr{R}(\mathscr{A}/\mathscr{K}))$. Thus, the components of $\mathscr{S}$ are the sets of semi-Fredholm operators $\mathscr{F}_\alpha$ for each fixed α $(-\infty\leqq\alpha\leqq\infty)$. We use the notation

$$\mathscr{F}^\alpha = \bigcup\{\mathscr{F}_\beta, \beta\neq\alpha, -\infty\leqq\beta\leqq\infty\}.$$

The following is our main result.

THEOREM 3. *The uniform closure of $\mathscr{F}_\alpha$ in a factor $\mathscr{A}$ of type II_∞ on a separable Hilbert space is the complement of $\mathscr{F}^\alpha$ in $\mathscr{A}$.*

PROOF. It must be shown that

$$\bar{\mathscr{F}}_\alpha = \mathscr{A} - \mathscr{F}^\alpha, \qquad -\infty\leqq\alpha\leqq\infty.$$

But $\mathscr{F}\subset_\alpha\mathscr{A}-\mathscr{F}^\alpha$, and $\mathscr{A}-\mathscr{F}^\alpha$ is closed. Thus, $\bar{\mathscr{F}}_\alpha\subset\mathscr{A}-\mathscr{F}^\alpha$. We prove the reverse inclusion.

We treat first the case $\alpha=0$. If $A \in \mathscr{G}(\mathscr{A})$, then A is Fredholm relative to $\mathscr{A}$ of index zero. Hence, $\mathscr{G}(\mathscr{A})\subset\mathscr{F}_0$, and $\mathrm{Cl}(\mathscr{G}(\mathscr{A}))\subset\overline{\mathscr{F}}_0$. By Corollary 1, the complement of $\mathrm{Cl}(\mathscr{G}(\mathscr{A}))$ in $\mathscr{A}$-consists of those operators of $\mathscr{A}$ that are semi-Fredholm relative to $\mathscr{A}$ of index other than zero. Thus, $\mathscr{A}-\mathrm{Cl}(\mathscr{G}(\mathscr{A}))\subset\mathscr{F}^0$, so that $\mathscr{A}-\mathscr{F}^0\subset\mathrm{Cl}(\mathscr{G}(\mathscr{A}))\subset\overline{\mathscr{F}}_0$, and $\mathscr{A}-\mathscr{F}^0=\overline{\mathscr{F}}_0$, proving the theorem for $\alpha=0$.

Let $\mathscr{G}_1$ denote the component of the identity in $\mathscr{G}(\mathscr{A}/\mathscr{K})$, and let $\mathscr{C}$ be an element of $H(\mathscr{G}(\mathscr{A}/\mathscr{K}))$. Since $\mathscr{C}=g\mathscr{G}_1$ for some $g\in\mathscr{G}(\mathscr{A}/\mathscr{K})$, and since multiplication by g is a homeomorphism, $\overline{\mathscr{C}}=g\overline{\mathscr{G}}_1$. By Theorem 1, $\mathrm{Cl}(\pi^{-1}(\mathscr{C}))=\pi^{-1}(\overline{\mathscr{C}})$ for each $\mathscr{C}$ in $H(\mathscr{G}(\mathscr{A}/\mathscr{K}))$. Therefore, by Proposition 1,

$$\mathscr{F}_0 \subset \pi^{-1}(\overline{\mathscr{G}}_1) = \overline{\pi^{-1}(\mathscr{G}_1)} \subset \mathscr{A} - \mathscr{F}^0 = \overline{\mathscr{F}}_0,$$

and since $\pi^{-1}(\overline{\mathscr{G}}_1)$ is closed, $\pi^{-1}(\overline{\mathscr{G}}_1)=\overline{\mathscr{F}}_0$. It follows that

$$\overline{\mathscr{G}}_1 = \mathscr{G}_1 \cup (\mathscr{A}/\mathscr{K} - (\mathscr{G} \cup \mathscr{L} \cup \mathscr{R})),$$

and

$$\begin{aligned} g\overline{\mathscr{G}}_1 = \overline{\mathscr{C}} &= g\mathscr{G}_1 \cup (\mathscr{A}/\mathscr{K} - (\mathscr{G} \cup \mathscr{L} \cup \mathscr{R})) \\ &= \mathscr{C} \cup (\mathscr{A}/\mathscr{K} - (\mathscr{G} \cup \mathscr{L} \cup \mathscr{R})). \end{aligned}$$

Thus by Theorem 1,

$$\overline{\mathscr{F}}_\alpha = \overline{\pi^{-1}(\mathscr{C})} = \pi^{-1}(\overline{\mathscr{C}}) = \mathscr{A} - \mathscr{F}^\alpha,$$

which is the desired result.

COROLLARY 2. *If $\mathscr{A}$ is a factor of type II_∞ on a separable Hilbert space, then the Breuer index has no continuous extension to $\mathscr{A}-\mathscr{S}$.*

PROOF. If $A\in\mathscr{A}-\mathscr{S}$ then each neighborhood of A contains Fredholm operators relative to $\mathscr{A}$ of every index.

Consider the C^*-algebra $\mathscr{A}/\mathscr{K}$. If s is a continuous cross section of $\mathscr{A}/\mathscr{K}$ into $\mathscr{A}$, and if i is the Breuer index on $\mathscr{F}$, then $i'=i\circ s$ is an index on $\mathscr{A}/\mathscr{K}$. If $x\notin\mathscr{G}(\mathscr{A}/\mathscr{K})$, then $s(x)\in\mathscr{A}-\mathscr{F}$. Hence, every neighborhood of $s(x)$ contains Fredholm operators relative to $\mathscr{A}$ of every finite index. By the continuity of π, $x=\pi(s(x))$ contains invertibles of $\mathscr{A}/\mathscr{K}$ of every finite index. We have proved the following.

THEOREM 4. *If i' is the index on $\mathscr{G}(\mathscr{A}/\mathscr{K})$ induced by the Breuer index on $\mathscr{F}$, then i' has no continuous extension to $\mathscr{A}/\mathscr{K}-\mathscr{G}(\mathscr{A}/\mathscr{K})$.*

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Department of Mathematics, Bernard M. Baruch College, City University of New York, New York, New York 10010