EXTENSIONS OF THE INDEX IN FACTORS OF TYPE $\Pi_\infty$

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ABSTRACT. In this paper we show that the analytic index has no continuous extension to those operators in a factor of type $\Pi_\infty$ on a separable Hilbert space which are not semi-Fredholm in the Breuer sense. A similar result has already been proved by Coburn and Lebow [3] for factors of type $I_\infty$. Here we use Breuer’s generalized Fredholm theory to extend their result to the more general setting.

1. Definitions and preliminaries. As usual, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on the separable Hilbert space $\mathcal{H}$. A *-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology is called a von Neumann algebra. If the center of $\mathcal{A}$ consists precisely of scalar multiples of the identity, then $\mathcal{A}$ is called a factor. For $E, F$ in $\mathcal{P}(\mathcal{A})$, the set of all projection operations in $\mathcal{A}$, we write $E \leq F \leftrightarrow EF = E$. The equivalence relation $\sim$ on $\mathcal{P}(\mathcal{A})$ is defined by $E \sim F$ if and only if there is a partially isometric operator $U$ in $\mathcal{A}$ such that $E = U^*U$ and $F = UU^*$. Finally, an order relation $\preceq$ on $\mathcal{P}(\mathcal{A})$ is given by $E \preceq F$ if and only if there is an $F'$ in $\mathcal{P}(\mathcal{A})$ such that $E \sim F' \leq F$.

A projection operator $E$ is said to be finite if it is not equivalent to any $F \in \mathcal{P}(\mathcal{A})$ where $F \leq E$ and $F \neq E$. Otherwise, $E$ is said to be infinite. If the identity of a von Neumann algebra $\mathcal{A}$ is a finite (infinite) projection, then $\mathcal{A}$ is called finite (infinite).

We follow Breuer’s generalization of the concepts of compact and Fredholm operators to a von Neumann algebra $\mathcal{A}$. For $B \in \mathcal{A}$

$$N_B = \sup\{E \in \mathcal{P}(\mathcal{A}) : BE = 0\}$$

and

$$R_B = \inf\{E \in \mathcal{P}(\mathcal{A}) : EB = B\}$$

are called the null projection and range projection of $B$, respectively. We call $B$ finite if $R_B$ is a finite projection. If $\mathcal{H}$ is the norm closure of the

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set of finite elements of \( \mathcal{A} \), then \( \mathcal{K} \) is a closed two-sided ideal in \( \mathcal{A} \). The elements of \( \mathcal{K} \) are said to be compact relative to \( \mathcal{A} \).

**Definition.** Let \( \mathcal{A} \) be a factor. An operator \( B \in \mathcal{A} \) is said to be Fredholm relative to \( \mathcal{A} \) if

(i) \( NB \) is a finite projection, and

(ii) there is a finite projection \( E \) in \( \mathcal{A} \) such that the range of \( I - E \) is contained in the range of \( B \).

Let \( \mathcal{F}(\mathcal{A}) \) be the set of Fredholm elements relative to a factor \( \mathcal{A} \), and let \( \text{Dim} \) denote some fixed relative dimension function on \( \mathcal{A} \) [6]. It follows from the definitions that if \( B \) is Fredholm relative to \( \mathcal{A} \) then \( \text{Dim} N_B \) and \( \text{Dim} NB \) are both finite. We can thus define the index, \( i \), of a Fredholm element \( B \) relative to \( \mathcal{A} \) by

\[
  i(B) = \text{Dim} N_B - \text{Dim} NB^*.
\]

The classical Fredholm theory is generalized by Breuer to compact and Fredholm operators relative to a factor. Specifically, if \( \mathcal{A} \) is an infinite factor, then \( B \in \mathcal{A} \) is Fredholm relative to \( \mathcal{A} \) if and only if \( \pi(B) \) is invertible in \( \mathcal{A}/\mathcal{K} \); for \( A, B \in \mathcal{F}(\mathcal{A}) \), \( i(A) = i(B) \) if and only if \( A \) and \( B \) lie in the same connected component of \( \mathcal{F}(\mathcal{A}) \) [2].

Let \( \mathcal{G}(\mathcal{A}/\mathcal{K}), \mathcal{L}(\mathcal{A}/\mathcal{K}), \text{and } \mathcal{H}(\mathcal{A}/\mathcal{K}) \) denote the open semigroups of invertible, left (but not right), and right (but not left) invertible elements of \( \mathcal{A}/\mathcal{K} \), respectively. The elements of

\[
  \mathcal{S} = \pi^{-1}(\mathcal{G}(\mathcal{A}/\mathcal{K})) \cup \pi^{-1}(\mathcal{L}(\mathcal{A}/\mathcal{K})) \cup \pi^{-1}(\mathcal{H}(\mathcal{A}/\mathcal{K}))
\]

are called semi-Fredholm relative to \( \mathcal{A} \), and the notation \( H(\mathcal{S}) \) and \( H(\pi(\mathcal{S})) \) is used to denote the set of connected components of \( \mathcal{S} \) and \( \pi(\mathcal{S}) \), respectively.

We shall use the following in the proof of our main result.

**Proposition 1.** The map \( \pi: H(\mathcal{S}) \to H(\pi(\mathcal{S})) \) defined by \( \pi: C \to \pi(C) \) for \( C \) in \( H(\mathcal{S}) \) is an isomorphism.

**Proof.** [3].

Essential in what follows is the notion of a cross section. Suppose \( f \) is a continuous mapping of \( X \) onto \( Y \), where \( X \) and \( Y \) are Banach spaces. A continuous map \( s: Y \to X \) is a continuous cross section of \( f \) if \( f(s(y)) = y \) for all \( y \in Y \). A consequence of the theorem of Bartle and Graves [1] is that if \( \pi \) is the projection of \( \mathcal{A} \) onto \( \mathcal{A}/\mathcal{K} \), then \( \pi \) has a continuous cross section. An immediate result of this is the following.

**Theorem 1.** Let \( C \) be a component of \( H(\mathcal{G}(\mathcal{A}/\mathcal{K})) \). Then \( \text{Cl}({\pi^{-1}(C)}) = \pi^{-1}(\pi(C)) \).
Proof. Since $\pi^{-1}(C)$ is a closed set containing $\pi^{-1}(C)$, $\text{Cl}(\pi^{-1}(C)) \subset \pi^{-1}(C)$. On the other hand, if $x \in \pi^{-1}(C)$, then $\pi(x) \in C$. Suppose $\{y_n\}$ in $C$ is such that $y_n \to \pi(x)$. If $s$ is a continuous cross section of $\pi$, then $s(y_n) \to s(\pi(x)) = x + k$ for some $k \in \mathcal{H}$. But then, $s(y_n) - k \to x$, and $s(y_n) = k \in \pi^{-1}(C)$. Hence, $x \in \text{Cl}(\pi^{-1}(C))$, and $\pi^{-1}(C) = \text{Cl}(\pi^{-1}(C))$.

Definition. A factor $\mathcal{A}$ is said to be of type $\Pi_\infty$ if the range of $\text{Dim}$ on $\mathcal{P}(\mathcal{A})$ is the interval $[0, \infty]$.

2. Main results. In the following, we denote by $r(A)$ the closure of the range of the operator $A$. We use the same notation to indicate a closed subspace of $\mathcal{H}$ and the orthogonal projection on this subspace.

We need the following result of Feldman and Kadison:

Proposition 2. If $\mathcal{A}$ is a factor on a separable Hilbert space, then $\text{Cl}(\mathcal{G}(\mathcal{A}))$ consists of those $A \in \mathcal{A}$ such that for every $\varepsilon > 0$ there is a projection $E \in \mathcal{A}$ containing the null projection of $A$ with $\|AE\| < \varepsilon$ and $E \sim \mathcal{H} \ominus r(A(I - E))$.

Proof. [5].

Feldman and Kadison proceed to prove that if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ then $A \notin \text{Cl}(\mathcal{G}(\mathcal{A}))$ if and only if $A$ is the product of a regular operator and a partially isometric operator between subspaces of unequal codimension. For factors of type $\Pi_\infty$, we get the following modified result.

Theorem 2. If $\mathcal{A}$ is a factor of type $\Pi_\infty$ on a separable Hilbert space, and if $A \in \mathcal{A}$ is such that $A \notin \text{Cl}(\mathcal{G}(\mathcal{A}))$, then $A$ is the product of a Fredholm operator relative to $\mathcal{A}$ of index zero and a partial isometry between subspaces of unequal relative codimension.

Proof. Suppose that $A \notin \text{Cl}(\mathcal{G}(\mathcal{A}))$, and let $N$ be the null projection of $A$. We have that $\dim N \neq \dim \mathcal{H} \ominus r(A)$, for otherwise, $N \sim \mathcal{H} \ominus r(A)$, and choosing $N$ as the $E$ of Proposition 2, $A \in \text{Cl}(\mathcal{G}(\mathcal{A}))$. We can assume that $\dim N < \dim \mathcal{H} \ominus r(A)$, for, if not, we can deal with $A^*$ since $N = \mathcal{H} \ominus r(A^*)$, and $N_{A^*} = \mathcal{H} \ominus r(A)$. Hence, $\dim N < \infty$, and $N$ is a finite projection.

If $A = U(A^*A)^{1/2}$ is the polar factorization of $A$ [4], then $U$ is a partial isometry mapping $r[(A^*A)^{1/2}] = r(A^*A) = \mathcal{H} \ominus N$ onto $r(A)$. Thus, the relative codimensions of the initial and final spaces of $U$ are unequal. It remains to be shown that $T = (A^*A)^{1/2}$ is a Fredholm operator relative to $\mathcal{A}$ of index zero.

Since $N = N_T$ has finite relative dimension, it suffices to show that there exists a finite projection $E$ such that the range of $I - E$ is contained in the range of $T$. Since $A \notin \text{Cl}(\mathcal{G}(\mathcal{A}))$, there exists, by Proposition 2, a constant $k > 0$ such that if $E$ is a projection in $\mathcal{A}$ with $\|AE\| \leq k$, then $E$ is not
equivalent to $\mathcal{H} \ominus r[A(I-E)]$. If $E$ is the spectral projection for $T$ corresponding to the interval $[0, k]$, then $E \in \mathcal{A}$ [4], and $\|AE\| = \|UTE\| \leq \|U\| \|TE\| \leq k$. Since the subspace $E$ contains $N_T$ [2], $E$ is not equivalent to $\mathcal{H} \ominus r[A(I-E)]$. We claim that $E$ is a finite projection. Assume that $E$ is infinite. Since $T(I-E) \geq k(I-E)$, the operator $T$ maps the space $I-E$ onto itself. Furthermore, we have that

$$r[U(I-E)] = r[UT(I-E)] = r[A(I-E)].$$

But if $E$ is infinite and $N$ finite, the space $r[U(E-N)]$ is infinite. Since $N$ is contained in $E$, $I-E$ is orthogonal to $E-N$, and because $U$ is an isometry on $\mathcal{H} \ominus N$, $r[U(E-N)]$ is also orthogonal to $r[U(I-E)]$. Hence, $\mathcal{H} \ominus r[U(I-E)] = \mathcal{H} \ominus r[A(I-E)]$ is infinite. Since $\mathcal{H}$ is separable, it follows that $E$ is equivalent to $\mathcal{H} \ominus r[A(I-E)]$, which is impossible. Hence, $E$ is finite. We also have that $r(I-E) \subset r(T)$ [2]. Therefore, $T$ is Fredholm relative to $\mathcal{A}$, and since $T$ is positive, its index is zero.

**Corollary 1.** If $A \in \mathcal{A}$ satisfies the hypothesis of the theorem, then $A$ is semi-Fredholm relative to $\mathcal{A}$ with $i(A) \neq 0$.

**Proof.** By the theorem, $A$ is the product of a partial isometry $U$ between subspaces of unequal relative codimension and a Fredholm operator relative to $\mathcal{A}$ of index zero. Since $\mathcal{H}$ is separable, one of these subspaces has finite relative codimension. Therefore, $U$ is semi-Fredholm relative to $\mathcal{A}$ with $i(U) \neq 0$, and the same must be true for $A$.

Let $\mathcal{S}$ be the set of semi-Fredholm operators in a factor $\mathcal{A}$ of type $\Pi_\infty$ on a separable Hilbert space. Noting for $A \in \pi^{-1}(\mathcal{L}(\mathcal{A} | \mathcal{H}))$ (resp. $\pi^{-1}(\mathcal{R}(\mathcal{A} | \mathcal{H}))$) that $\dim N_A < \infty$ (resp. $\dim N_A \cdot \infty < \infty$) [2], we write $\mathcal{F}_\infty = \pi^{-1}(\mathcal{L}(\mathcal{A} | \mathcal{H}))$ and $\mathcal{F}_{-\infty} = \pi^{-1}(\mathcal{R}(\mathcal{A} | \mathcal{H}))$. Thus, the components of $\mathcal{S}$ are the sets of semi-Fredholm operators $\mathcal{F}_\alpha$ for each fixed $\alpha$ ($-\infty \leq \alpha \leq \infty$). We use the notation

$$\mathcal{F}_\alpha = \bigcup \{ \mathcal{F}_\beta, \beta \neq \alpha, -\infty \leq \beta \leq \infty \}.$$

The following is our main result.

**Theorem 3.** The uniform closure of $\mathcal{F}_\alpha$ in a factor $\mathcal{A}$ of type $\Pi_\infty$ on a separable Hilbert space is the complement of $\mathcal{F}_\alpha$ in $\mathcal{A}$.

**Proof.** It must be shown that

$$\mathcal{F}_\alpha = \mathcal{A} - \mathcal{F}_\alpha, \quad -\infty \leq \alpha \leq \infty.$$

But $\mathcal{F} \subset_{\alpha} \mathcal{A} - \mathcal{F}_\alpha$, and $\mathcal{A} - \mathcal{F}_\alpha$ is closed. Thus, $\mathcal{F}_\alpha \subset \mathcal{A} - \mathcal{F}_\alpha$. We prove the reverse inclusion.
We treat first the case \(\alpha=0\). If \(A \in \mathcal{G}(\mathcal{A})\), then \(A\) is Fredholm relative to \(\mathcal{A}\) of index zero. Hence, \(\mathcal{G}(\mathcal{A}) \subseteq \mathcal{F}_0\), and \(\text{Cl}(\mathcal{G}(\mathcal{A})) \subseteq \mathcal{F}_0\). By Corollary 1, the complement of \(\text{Cl}(\mathcal{G}(\mathcal{A}))\) in \(\mathcal{A}\)-consists of those operators of \(\mathcal{A}\) that are semi-Fredholm relative to \(\mathcal{A}\) of index other than zero. Thus, \(\mathcal{A} - \text{Cl}(\mathcal{G}(\mathcal{A})) \subseteq \mathcal{F}_0\), so that \(\mathcal{A} - \mathcal{F}_0 \subseteq \text{Cl}(\mathcal{G}(\mathcal{A})) \subseteq \mathcal{F}_0\), and \(\mathcal{A} - \mathcal{F}_0 = \mathcal{F}_0\), proving the theorem for \(\alpha=0\).

Let \(\mathcal{G}_1\) denote the component of the identity in \(\mathcal{G}(\mathcal{A}/\mathcal{H})\), and let \(\mathcal{E}\) be an element of \(H(\mathcal{G}(\mathcal{A}/\mathcal{H}))\). Since \(\mathcal{E}=g\mathcal{G}_1\) for some \(g \in \mathcal{G}(\mathcal{A}/\mathcal{H})\), and since multiplication by \(g\) is a homeomorphism, \(\mathcal{E}=g\mathcal{G}_1\). By Theorem 1, \(\text{Cl}(\pi^{-1}(\mathcal{E})) = \pi^{-1}(\mathcal{E})\) for each \(\mathcal{E}\) in \(H(\mathcal{G}(\mathcal{A}/\mathcal{H}))\). Therefore, by Proposition 1,

\[
\mathcal{F}_0 \subseteq \pi^{-1}(\mathcal{G}_1) = \pi^{-1}(\mathcal{G}_1) \subseteq \mathcal{A} - \mathcal{F}_0 = \mathcal{F}_0,
\]

and since \(\pi^{-1}(\mathcal{G}_1)\) is closed, \(\pi^{-1}(\mathcal{G}_1) = \mathcal{F}_0\). It follows that

\[
\mathcal{G}_1 = \mathcal{G}_1 \cup (\mathcal{A}/\mathcal{H} - (\mathcal{G} \cup \mathcal{L} \cup \mathcal{R})),
\]

and

\[
g\mathcal{G}_1 = g\mathcal{G}_1 \cup (\mathcal{A}/\mathcal{H} - (\mathcal{G} \cup \mathcal{L} \cup \mathcal{R}))
= g \cup (\mathcal{A}/\mathcal{H} - (\mathcal{G} \cup \mathcal{L} \cup \mathcal{R})).
\]

Thus by Theorem 1,

\[
\mathcal{F}_2 = \pi^{-1}(\mathcal{E}) = \pi^{-1}(\mathcal{E}) = \mathcal{A} - \mathcal{F}^2,
\]

which is the desired result.

**Corollary 2.** If \(\mathcal{A}\) is a factor of type \(\Pi_\infty\) on a separable Hilbert space, then the Breuer index has no continuous extension to \(\mathcal{A} - \mathcal{F}\).

**Proof.** If \(A \in \mathcal{A} - \mathcal{F}\) then each neighborhood of \(A\) contains Fredholm operators relative to \(\mathcal{A}\) of every index.

Consider the C*-algebra \(\mathcal{A}/\mathcal{H}\). If \(s\) is a continuous cross section of \(\mathcal{A}/\mathcal{H}\) into \(\mathcal{A}\), and if \(i\) is the Breuer index on \(\mathcal{F}\), then \(i' = i \circ s\) is an index on \(\mathcal{A}/\mathcal{H}\). If \(x \notin \mathcal{G}(\mathcal{A}/\mathcal{H})\), then \(s(x) \in \mathcal{A} - \mathcal{F}\). Hence, every neighborhood of \(s(x)\) contains Fredholm operators relative to \(\mathcal{A}\) of every finite index. By the continuity of \(i\), \(x = \pi(s(x))\) contains invertibles of \(\mathcal{A}/\mathcal{H}\) of every finite index. We have proved the following.

**Theorem 4.** If \(i'\) is the index on \(\mathcal{G}(\mathcal{A}/\mathcal{H})\) induced by the Breuer index on \(\mathcal{F}\), then \(i'\) has no continuous extension to \(\mathcal{A}/\mathcal{H} - \mathcal{G}(\mathcal{A}/\mathcal{H})\).

**References**


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