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COMMUTANTS AND CYCLIC VECTORS

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ABSTRACT. This note analyzes the relationship between various statements concerning the commutant of a bounded linear operator on a Hilbert space and the existence of cyclic vectors for the operator and its adjoint.

For a bounded linear operator T on a Hilbert space H, $\mathscr{A}_T$ will denote the weakly closed algebra generated by T and the identity I, and $\mathscr{A}'_T$ will denote the commutant of T. We say that T has a cyclic vector x_0 if $\{\mathscr{A}_T x_0\}$ is dense in H. For a finite dimensional operator or a normal operator the following are known to be equivalent (see [**1**]):

(M_1) T and T^* have cyclic vectors.

(M_2) T or T^* has a cyclic vector.

(M_3) $\mathscr{A}'_T$ is abelian.

In this note we analyze the relationships (this was asked by Professor B. Sz.-Nagy at the International Conference on Operator Theory, Indiana University, 1970) between the above statements and the following statements (which are also equivalent for finite dimensional operators):

(M_4) $\mathscr{A}_T=\mathscr{A}'_T$.

(M_5) Every invariant subspace for T is hyperinvariant (i.e., invariant for $\mathscr{A}'_T$).

EXAMPLE 1. Let T be the bilateral shift on l^2 defined by $Te_n=e_{n+1}$. Then T satisfies M_1, M_2, M_3 but not M_4, M_5 (see [**3**, Problems 115, 116], [**4**]).

EXAMPLE 2 (DUE TO D. SARASON). Let S be the unilateral shift on l^2_+ defined by $Se_n=e_{n+1}$ $(n\geqq 0)$, and $T=S^*\oplus S^*$. Then T satisfies M_2 but not M_1, M_3, M_4, M_5 (see [**3**, Problem 126], [**4**]).

EXAMPLE 3. Let B be the bilateral weighted shift on l^2 defined by $Be_n=2^n e_{n+1}$ $(n<0)$ and $Be_n=e_{n+1}$ $(n\geqq 0)$. Then B is not invertible and hence $\mathscr{A}_B=\mathscr{A}'_B$ [**2**], [**7**]. Moreover, $\Pi_0(B^*)=\{z:0<|z|<1\}$. Therefore B^* has a cyclic vector, but B does not [**6**]. Let $T=(B+2)\oplus(B^*-2)$. The

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fact that the spectrum of T is the disjoint union of the two discs comprising the spectrum of $B+2$ and B^*-2 implies that [5, §5].

$$\mathscr{A}'_T = \mathscr{A}'_{B+2} \oplus \mathscr{A}'_{B^*-2} = \mathscr{A}_{B+2} \oplus \mathscr{A}_{B^*-2} = \mathscr{A}_T.$$

If (x, y) were a cyclic vector for T (resp. T^*), then x (resp. y) would be cyclic for B. T satisfies M_3, M_4, M_5 but not M_1, M_2.

The implications $M_1 \Rightarrow M_2$, $M_4 \Rightarrow M_3$, and $M_4 \Rightarrow M_5$ are trivially true, and the Examples 1, 2, 3 show that any other implication is false, except perhaps

(1) Does M_1 imply M_3?

(2) (Due to Douglas and Pearcy [1]) Does M_5 imply M_3 or M_4?

which are still open problems.

We close by remarking that Example 3 answers in the negative the question raised by the first author at the Conference on Operator Theory, Durham, 1971. That question was: Does $\mathscr{A}'_T$ always have a cyclic vector? Our answer: Not necessarily, even when $\mathscr{A}'_T = \mathscr{A}_T$.

References

1. R. G. Douglas and C. Pearcy, *On a topology for invariant subspaces*, J. Functional Analysis **2** (1968), 323–341. MR **38** #1547.

2. R. Gellar, *Two sublattices of weighted shift invariant subspaces*, Indiana Univ. Math. J. **23** (1973), 1–10.

3. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

4. H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964. MR **30** #1409.

5. D. A. Herrero and N. Salinas, *nalytically invariant and biinvariant subspaces*, Trans. Amer. Math. Soc. **173** (1972), 117–136.

6. D. A. Herrero, *Eigenvectors and cyclic vectors for bilateral weighted shifts*, Rev. Un. Mat. Argentina. **26** (1972), 24–41.

7. T. R. Turner, *Double commutants of singly generated operator algebras*, Dissertation, University of Michigan, Ann Arbor, Mich., 1971.

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