

COMMUTANTS AND CYCLIC VECTORS

JAMES A. DEDDENS,¹ RALPH GELLAR² AND DOMINGO A. HERRERO³

ABSTRACT. This note analyzes the relationship between various statements concerning the commutant of a bounded linear operator on a Hilbert space and the existence of cyclic vectors for the operator and its adjoint.

For a bounded linear operator T on a Hilbert space H , \mathcal{A}_T will denote the weakly closed algebra generated by T and the identity I , and \mathcal{A}'_T will denote the commutant of T . We say that T has a cyclic vector x_0 if $\{\mathcal{A}_T x_0\}$ is dense in H . For a finite dimensional operator or a normal operator the following are known to be equivalent (see [1]):

- (M₁) T and T^* have cyclic vectors.
- (M₂) T or T^* has a cyclic vector.
- (M₃) \mathcal{A}'_T is abelian.

In this note we analyze the relationships (this was asked by Professor B. Sz.-Nagy at the International Conference on Operator Theory, Indiana University, 1970) between the above statements and the following statements (which are also equivalent for finite dimensional operators):

- (M₄) $\mathcal{A}_T = \mathcal{A}'_T$.
- (M₅) Every invariant subspace for T is hyperinvariant (i.e., invariant for \mathcal{A}'_T).

EXAMPLE 1. Let T be the bilateral shift on l^2 defined by $Te_n = e_{n+1}$. Then T satisfies M₁, M₂, M₃ but not M₄, M₅ (see [3, Problems 115, 116], [4]).

EXAMPLE 2 (DUE TO D. SARASON). Let S be the unilateral shift on l^2_+ defined by $Se_n = e_{n+1}$ ($n \geq 0$), and $T = S^* \oplus S^*$. Then T satisfies M₂ but not M₁, M₃, M₄, M₅ (see [3, Problem 126], [4]).

EXAMPLE 3. Let B be the bilateral weighted shift on l^2 defined by $Be_n = 2^n e_{n+1}$ ($n < 0$) and $Be_n = e_{n+1}$ ($n \geq 0$). Then B is not invertible and hence $\mathcal{A}_B = \mathcal{A}'_B$ [2], [7]. Moreover, $\Pi_0(B^*) = \{z: 0 < |z| < 1\}$. Therefore B^* has a cyclic vector, but B does not [6]. Let $T = (B+2) \oplus (B^*-2)$. The

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fact that the spectrum of T is the disjoint union of the two discs comprising the spectrum of $B+2$ and B^*-2 implies that [5, §5].

$$\mathcal{A}'_T = \mathcal{A}'_{B+2} \oplus \mathcal{A}'_{B^*-2} = \mathcal{A}_{B+2} \oplus \mathcal{A}_{B^*-2} = \mathcal{A}_T.$$

If (x, y) were a cyclic vector for T (resp. T^*), then x (resp. y) would be cyclic for B . T satisfies M_3, M_4, M_5 but not M_1, M_2 .

The implications $M_1 \Rightarrow M_2$, $M_4 \Rightarrow M_3$, and $M_4 \Rightarrow M_5$ are trivially true, and the Examples 1, 2, 3 show that any other implication is false, except perhaps

(1) Does M_1 imply M_3 ?

(2) (Due to Douglas and Pearcy [1]) Does M_5 imply M_3 or M_4 ?

which are still open problems.

We close by remarking that Example 3 answers in the negative the question raised by the first author at the Conference on Operator Theory, Durham, 1971. That question was: Does \mathcal{A}'_T always have a cyclic vector? Our answer: Not necessarily, even when $\mathcal{A}'_T = \mathcal{A}_T$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66044

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607

Current address (Domingo A. Herrero): Universidad Nacional de Rio IV, Provincia de Córdoba, Republica Argentina