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A NONSTATIONARY ITERATIVE PROCESS FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. It is shown that a nonstationary analogue of an iterative process of Kirk serves to approximate fixed points of compact nonexpansive mappings defined on convex subsets of a uniformly convex space.

In a recent note Kirk [1] investigated an iterative process for approximating fixed points of nonexpansive mappings defined on convex subsets of a uniformly convex Banach space. A mapping T is called *nonexpansive* if $\|Tx-Ty\| \leqq \|x-y\|$ for each x and y in the domain of T.

Specifically, the iterative process studied by Kirk is given by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \cdots + \alpha_k T^k x_n \tag{1}$$

where $\alpha_i \geqq 0$, $\alpha_1 > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$.

It is the purpose of this note to show that a nonstationary analogue of the process (1) also serves to approximate fixed points of T under certain circumstances. We will make use of the fact that in a uniformly convex space with modulus of convexity δ, for given $\varepsilon > 0$, $d > 0$ and $\alpha \in [0, 1]$ the inequalities

$$\|w\| \leqq \|u\| \leqq d \quad \text{and} \quad \|u - w\| \geqq \varepsilon$$

imply that

$$\|(1-\alpha)u + \alpha w\| \leqq \|u\| \, [1 - 2\delta(\varepsilon/d)\min(\alpha, 1-\alpha)]$$

(see e.g. [2, p. 4]).

LEMMA 1. *If $\{u_n\}$ and $\{w_n\}$ are sequences in a uniformly convex space with $\|w_n\| \leqq \|u_n\|$ and*

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n w_n \qquad (0 \leqq \alpha_n \leqq 1)$$

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where $\sum \min(\alpha_n, 1-\alpha_n)=\infty$, *then* $0 \in \mathrm{cl}\{u_n - w_n\}$ (cl A *denotes the* (*strong*) *closure of the set* A).

PROOF. Suppose $\|u_n - w_n\| \geqq \varepsilon$ for all n, then

$$\begin{aligned} \|u_{n+1}\| &= \|(1-\alpha_n)u_n + \alpha_n w_n\| \\ &\leqq \|u_n\| \, [1 - 2\delta(\varepsilon/\|u_1\|)\min(\alpha_n, 1-\alpha_n)]. \end{aligned}$$

Inductively we have

$$\|u_n\| \leqq \|u_1\| \prod_{i=1}^{n-1} [1 - 2\delta(\varepsilon/\|u_1\|)\min(\alpha_i, 1-\alpha_i)] \quad \text{for } n > 1.$$

But since $\sum \min(\alpha_i, 1-\alpha_i)=\infty$, the product on the right diverges to zero and hence $\lim\|u_n\| = \lim\|w_n\| = 0$ and this contradiction completes the proof.

Let the sequences $\{\alpha_{ij}\}_{i=0}^{\infty}$ $(j=0, 1, \cdots, k)$ satisfy $0 \leqq \alpha_{ij}$, $0 < \alpha \leqq \alpha_{i1}$, $\sum_{j=0}^{k} \alpha_{ij} = 1$ for each i and $\sum \min(\alpha_{i0}, 1-\alpha_{i0})=\infty$. Define operators S_i by

$$S_i = \alpha_{i0} I + \alpha_{i1} T + \cdots + \alpha_{ik} T^k, \qquad i = 0, 1, 2, \cdots.$$

A nonstationary analogue of (1) is the process

$$x_{n+1} = S_n x_n, \qquad n = 0, 1, 2, \cdots. \tag{2}$$

Note that if p is a fixed point of T then

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \sum_{j=0}^{k} \alpha_{nj}(T^j x_n - T^j p) \right\| \\ &\leqq \|x_n - p\| \end{aligned} \tag{3}$$

and hence to establish the convergence of $\{x_n\}$ to a fixed point p it is enough to show that some subsequence of $\{x_n\}$ converges to p.

LEMMA 2. *Let K be a convex subset of a uniformly convex space. If T is a nonexpansive mapping of K into itself which has at least one fixed point and $\{x_n\}$ is defined by* (2), *then* $0 \in \mathrm{cl}\{x_{n+1} - x_n\}$.

PROOF. Let p be a fixed point of T and let

$$u_n = x_n - p \quad \text{and} \quad w_n = \frac{1}{1-\alpha_{n0}} \sum_{j=1}^{k} \alpha_{nj}(T^j x_n - T^j p).$$

We then have

$$u_{n+1} = S_n x_n - p = \alpha_{n0} u_n + (1-\alpha_{n0}) w_n$$

and $\|w_n\| \leqq \|u_n\|$ since T is nonexpansive. Thus, by Lemma 1,

$0 \in \mathrm{cl}\{u_n - w_n\}$. Also

$$\|u_n - w_n\| = \left\| x_n - p - \frac{1}{1-\alpha_{n0}} \sum_{j=1}^{k} \alpha_{nj} T^j x_n + p \right\|$$

$$= \left\| x_n - \frac{1}{1-\alpha_{n0}} \sum_{j=0}^{k} \alpha_{nj} T^j x_n + \frac{\alpha_{n0}}{1-\alpha_{n0}} x_n \right\|$$

$$= \frac{1}{1-\alpha_{n0}} \|x_n - x_{n+1}\| \geqq \|x_n - x_{n+1}\|$$

and therefore there is a subsequence $\{x_{n_i}\}$ with $x_{n_i+1} - x_{n_i} \to 0$.

We now give a generalization of Kirk's result on strong convergence of the sequence $\{x_n\}$ defined by (1).

THEOREM. *Suppose, in addition to the hypotheses of Lemma* 2, *that T is compact. Then for each $x_1 \in K$ the sequence $\{x_n\}$ defined by* (2) *converges to a fixed point of T.*

PROOF. By Lemma 2 there is a subsequence $\{x_{n_i}\}$ with $x_{n_i+1} - x_{n_i} \to 0$. Since $\alpha_{n_i j} \in [0, 1]$ and $\alpha_{n_i 1} \geqq \alpha > 0$ we may assume by successively choosing subsequences $\{\alpha_{n_\nu j}\}$ of the sequences $\{\alpha_{n_i j}\}$ that $\lim_\nu \alpha_{n_\nu j} = \alpha_j \in [0, 1]$ where $\alpha_1 > 0$. Let

$$S = \alpha_0 I + \alpha_1 T + \cdots + \alpha_k T^k.$$

We then have

$$x_{n_\nu} - S x_{n_\nu} = x_{n_\nu} - S_{n_\nu} x_{n_\nu} + S_{n_\nu} x_{n_\nu} - S x_{n_\nu}$$

where $x_{n_\nu} - S_{n_\nu} x_{n_\nu} = x_{n_\nu} - x_{n_\nu+1} \to 0$.

If p is a fixed point of T then we have by use of (3)

$$\|T^j x_{n_\nu} - p\| = \|T^j x_{n_\nu} - T^j p\| \leqq \|x_{n_\nu} - p\| \leqq \|x_1 - p\|.$$

Hence if we set $M = \|x_1 - p\| + \|p\|$, then $\|T^j x_{n_\nu}\| \leqq M$ for all ν and each $=0, \cdots, k$. Therefore

$$\|S_{n_\nu} x_{n_\nu} - S x_{n_\nu}\| = \left\| \sum_{j=0}^{k} (\alpha_{n_\nu j} - \alpha_j) T^j x_{n_\nu} \right\|$$

$$\leqq M \sum_{j=0}^{k} |\alpha_{n_\nu j} - \alpha_j| \to 0 \quad \text{as } \nu \to \infty.$$

It follows that $x_{n_\nu} - S x_{n_\nu} \to 0$ as $\nu \to \infty$. Since T is compact $I - S$ maps closed bounded subsets into closed subsets (see Kirk's argument in the Corollary to Theorem 2 of [**1**]). By (3) $\mathrm{cl}\{x_n\}$ is bounded and closed and we have shown that $0 \in (I-S)\mathrm{cl}\{x_n\}$. Thus there is a $z \in \mathrm{cl}\{x_n\}$ with

$z - Sz = 0$. By [**1**, Theorem 1] z is a fixed point of T and since $z \in \mathrm{cl}\{x_n\}$ it follows from (3) that $x_n \to z$ as $n \to \infty$, completing the proof.

References

1. W. A. Kirk, *On successive approximations for nonexpansive mappings in Banach spaces*, Glasgow Math. J. **12** (1971), 6–9. MR **45** #7553.

2. Z. Opial, *Nonexpansive and monotone mappings in Banach spaces*, Lecture Notes 67–1, Division of Applied Mathematics, Brown University, Providence, R.I., 1967.

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