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AN EXTENSION OF A THEOREM OF EISENSTEIN

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ABSTRACT. In the present paper I obtain an extension of the so-called Eisenstein theorem which is proved by means of Riemann-Stieltjes integration.

THEOREM. *Let n be a positive integer $\geqq 2$. Let $p_1, p_2, \cdots, p_n$ be odd integers and relatively prime in pairs. Then*

$$\sum_{i=1}^{n} \sum_{x=1}^{(p_i-1)/2} \prod_{k=1; k\neq i}^{n} \left[\frac{p_k x}{p_i}\right] = \frac{1}{2^n} \prod_{k=1}^{n} (p_k - 1).$$

PROOF. The method of proof is Riemann-Stieltjes integration. Consider the integrals

$$\mathscr{T}_1 = \int_0^{1/2} [p_1 x]\,[p_2 x] \cdots [p_{n-1} x]\, d[p_n x],$$

$$\mathscr{T}_2 = \int_0^{1/2} [p_n x]\, d[p_1 x]\,[p_2 x] \cdots [p_{n-1} x].$$

First we observe that the greatest integer functions $[p_1 x]$, $[p_2 x], \cdots$, $[p_n x]$ have no discontinuities in common on the interval $0 < x \leqq \frac{1}{2}$ in view of the condition on the integers $p_1, p_2, \cdots, p_n$. Second, the discontinuities of $[p_i x]$, $i = 1, 2, \cdots, n$, on the interval $0 < x \leqq \frac{1}{2}$ are at $x = k/p_i$, $k = 1, 2, \cdots, \frac{1}{2}(p_i - 1)$, and the jump of $[p_i x]$ at each of these discontinuities is equal to 1. Hence the value of $\mathscr{T}_1$ is

$$\sum_{x=k/p_n} [p_1 x]\,[p_2 x] \cdots [p_{n-1} x], \qquad k = 1, 2, \cdots, \tfrac{1}{2}(p_n - 1)$$

or

$$\sum_{x=1}^{(p_n-1)/2} \left[\frac{p_1 x}{p_n}\right] \left[\frac{p_2 x}{p_n}\right] \cdots \left[\frac{p_{n-1} x}{p_n}\right].$$

Consider the second integral. The discontinuities of $[p_1 x]\,[p_2 x] \cdots [p_{n-1} x]$ are at $x = k/p_i$, $k = 1, 2, \cdots, \frac{1}{2}(p_i - 1)$, $i = 1, 2, \cdots, n-1$. Since there are no common discontinuities and the jumps of $[p_i x]$ at the discontinuities k/p_i are equal to 1, the jump of $[p_1 x]\,[p_2 x] \cdots [p_{n-1} x]$ at each

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k/p_i is equal to

$$\left[\frac{p_1k}{p_i}\right]\left[\frac{p_2k}{p_i}\right]\cdots\left[\frac{p_{i-1}k}{p_i}\right]\left[\frac{p_{i+1}k}{p_i}\right]\cdots\left[\frac{p_{n-1}k}{p_i}\right]$$

and so we obtain the value of the second integral

$$\mathscr{T}_2=\sum_{i=1}^{n-1}\sum_{x=1}^{(p_i-1)/2}\prod_{k=1;k\neq i}^{n}\frac{p_kx}{p_i}.$$

This proves the theorem completely.

REMARK. For the history of the so-called Eisenstein theorem ($n=2$ in our Theorem) and other generalizations we refer to Bruce C. Berndt's paper: *A generalization of a theorem of Gauss on* [x], presented at the Third Illinois Conference on Number Theory on April 7, 1973.

As is well known the so-called Eisenstein theorem which says: *If p and q are two odd distinct primes then*

$$\sum_{x=1}^{(q-1)/2}\left[\frac{px}{q}\right]+\sum_{x=1}^{(p-1)/2}\left[\frac{qx}{p}\right]=(1/4)(p-1)(q-1)$$

is very helpful in proving Gauss's quadratic reciprocity theorem.

Finally, thanks are due to the referee for pointing out to me a more general statement than first submitted.

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