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IDEALS I OF $R[X]$ FOR WHICH $R[X]/I$ IS R-PROJECTIVE

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ABSTRACT. A characterization is given of those ideals I of the polynomial ring $R[X]$ such that $R[X]/I$ is R-projective. It is also shown that a commutative ring R has the property "$R[X]/I$ R-projective implies I is a finitely generated ideal" if and only if R has only a finite number of idempotents.

Let R be a commutative unitary ring and let I be an ideal of the polynomial ring $R[X]$. What conditions on I are necessary and sufficient in order that $R[X]/I$ be a projective R-module? Which rings R have the property that from $R[X]/I$ R-projective it follows that I is a finitely generated ideal? These two questions are raised in **[BM]** and our purpose here is to settle them. In **[BM]** it is shown that when $R[X]/I$ is projective, I is finitely generated if and only if I is principal. In this case, $I=fR[X]$ where f is *almost quasi-monic*, meaning that there exist pairwise orthogonal idempotents $e_0, e_1, \cdots, e_n$ of R such that $e_i f$ is a monic polynomial of $e_iR[X]$ for $0\leqq i\leqq n$ and if $e=\sum_{i=0}^n e_i$, then $(1-e)f=0$. Therefore, the problem arises when I is not finitely generated and it is this case that our theorem treats.

A fact, which we shall use implicitly and which is recorded explicitly in **[BM]**, is that, for $R[X]/I$ R-projective and for P a prime ideal of R, $I_P=IR_P[X]$ is 0 or principal generated by a monic polynomial.

Another such fact is the following: Let J be an ideal of $R[X]$. Denote by $c(J)$ the ideal of R generated by those elements of R occurring as the coefficient of some element of J. If $R[X]/J$ is R-flat, then $c(J)$ is a *pure* ideal of R, that is, for each prime ideal P of R, $(c(J))_P=0$ or $(c(J))_P=R_P$ **[OR**, Corollary 1.3, p. 380]. Moreover, if $R[X]/J$ is R-projective, J is finitely generated if and only if $c(J)$ is finitely generated **[BM**, Theorem 3].

THEOREM. *The ring R has the property "$R[X]/I$ is R-projective implies I is finitely generated" if and only if R has only a finite number of*

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idempotents. Moreover, for an ideal I of $R[X]$ *which is not finitely generated,* $R[X]/I$ *is R-projective if and only if there exists an infinite set* $\{e_i\}_{i=0}^{\infty}$ *of idempotents in R such that* $(e_0)<(e_1)<\cdots$, *and a set* $e_0=f_0, f_1, \cdots$ *of almost quasi-monic polynomials in I such that* $I=(f_0, f_1, \cdots)$, $e_i f_{i+1}=f_i$, *and the leading coefficient of* f_{i+1} *is* $e_{i+1}-e_i$, *for each i.*

PROOF. Suppose that $R[X]/I$ is R-projective and I is not finitely generated. Consider the R-submodules of $R[X]$ defined by $F_n=R+RX+\cdots+RX^n$ and $I_n=I\cap F_n$ for each nonnegative integer n. We first show that F_n/I_n and hence I_n are finitely generated projective R-modules. Consider the exact sequence

$$0\to F_n/I_n\to R[X]/I\to C_n\to 0$$

where the maps are the natural ones and C_n denotes the cokernel. Since F_n/I_n is a finitely generated R-module and since $R[X]/I$ is a countably generated projective R-module, to prove that F_n/I_n is R-projective, it suffices, by virtue of [**J**, Lemma 2, p. 57], to prove that C_n is R-flat. For this, we look locally at primes of R. Thus, let P be a prime ideal of R. If $P\supseteq c(I)$, then $I_P=(I_n)_P=0$ and so $(F_n)_P/(I_n)_P\cong R_P+R_PX+\cdots+R_PX^n$ and $R_P[X]/I_P\cong R_P[X]$. It follows that $(C_n)_P\cong R_PX^{n+1}+R_PX^{n+2}+\cdots$ is a free R_P-module. If $P\not\supseteq c(I)$, then $I_P\neq 0$, and since $R[X]/I$ is R-projective, I_P is generated by a monic polynomial of nonnegative degree in $R_P[X]$. Thus, there exists a polynomial $f\in I$ such that the image of f in $R_P[X]$ generates I_P. By multiplying, if necessary, by an appropriate element of $R\backslash P$, we may assume that f is such that the leading coefficient of f is a unit in R_P. Under this assumption, if $s=\text{degree}(f)>n$, then $(I_n)_P=0$. Thus, $(F_n)_P/(I_n)_P\cong R_P+R_PX+\cdots+R_PX^n$, $R_P[X]/I_P\cong R_P+R_PX+\cdots+R_PX^s$, and $(C_n)_P$ is R_P-free. If $s\leqq n$, then our assumption tells us that $f\in I_n$, and in fact, $f, Xf, \cdots, X^{n-s}f\in I_n$. Moreover, $1, X, \cdots, X^{s-1}, f, Xf, \cdots, X^{n-s}f$ form a free basis for $(F_n)_P$ as an R_P-module. Since any polynomial in $(I_n)_P$ has degree $\geqq s$, it follows that $f, Xf, \cdots, X^{n-s}f$ generate $(I_n)_P$. Thus $(F_n)_P/(I_n)_P$ is a free R_P-module generated by the images of $1, X, \cdots, X^{s-1}$, $(F_n)_P/(I_n)_P\cong R_P[X]/I_P$, and in this case $(C_n)_P=0$. We conclude that F_n/I_n is R-projective. Therefore, the sequence of R-modules $0\to I_n\to F_n\to F_n/I_n\to 0$ splits and I_n is a finitely generated projective R-module. It follows, for example from [**OR**, Corollary 1.3, p. 380], that $c(I_n)=$ideal of R generated by the coefficients of elements of I of degree $\leqq n$ is principal generated by an idempotent. Let $c(I_n)=eR$, where e is an idempotent element of R, and let $J=I_nR[X]$.

CLAIM. $R[X]/J$ *is R-isomorphic to* $(1-e)R[X]\oplus eF_n/I_n$ *and therefore* $R[X]/J$ *is R-projective.*

PROOF OF CLAIM. We have $c(J)=c(I_n)=eR$, so $R[X]/J=(1-e)R[X]\oplus eR[X]/J$. Thus, we need only show that eF_n/I_n is R-isomorphic to $eR[X]/J$. Since $I_n=I\cap F_n$, $J\cap F_n=I_n$ and F_n/I_n is canonically embedded in $R[X]/J$. Thus, eF_n/I_n is a submodule of $eR[X]/J$ and if $e, e\theta, \cdots, e\theta^n$ denote the images of $e, eX, \cdots, eX^n$ in $eR[X]/J$, then these elements are in the submodule eF_n/I_n. Hence, to show $eF_n/I_n=eR[X]/J$, it will suffice to show that $e, e\theta, \cdots, e\theta^n$ generate the R-module $eR[X]/J$. We show this by showing $e, e\theta, \cdots, e\theta^n$ generate $eR[X]/J$ locally at any prime P of R. If $e\in P$, then $(eR[X]/J)_P=0$; and if $e\notin P$, then $c(I_n)\not\subseteq P$, so $J_P=I_P$ is generated by a monic polynomial in $R_P[X]$ of degree $\leqq n$. Thus, the images of $e, e\theta, \cdots, e\theta^n$ in $(eR_P[X]/J)$ generate $(eR_P[X]/J)_P$ as an R_P-module. We conclude that $eF_n/I_n=eR[X]/J$, which completes the proof of the claim.

We clearly have $\bigcup_{n=0}^{\infty} c(I_n)=c(I)$ and by assumption I, and therefore $c(I)$, is not finitely generated. Hence, the ascending sequence of ideals $c(I_0)\subseteqq c(I_1)\subseteqq\cdots$ does not become stationary and we may assume that the notation is as follows: $e_0R=c(I_0)=\cdots=c(I_{n_1-1})<e_1R=c(I_{n_1})=\cdots=c(I_{n_2-1})<e_2R=c(I_{n_2})=\cdots$, the e_i's idempotents of R and the n_i's a strictly ascending sequence of positive integers. Let $J_k=I_kR[X]$. Then J_k is a finitely generated ideal of $R[X]$ and we have shown above that $R[X]/J_k$ is R-projective. Hence by [**BM**, Theorem 3], J_k is a principal ideal of $R[X]$ generated by an almost quasi-monic polynomial. We observe that if $m\leqq n$ and $c(J_m)=c(J_n)$, then $J_m=J_n$. For if P is a prime of R and $c(J_m)=c(J_n)\subseteqq P$, then $(J_m)_P=(J_n)_P=0$; and if $c(J_m)\not\subseteq P$, then $(J_m)_P=I_P\supseteqq(J_n)_P$, so $(J_m)_P=(J_n)_P$. We denote the almost quasi-monic generator of J_{n_i} by f_i. Note that $c(f_i)=e_i$ and $J_{n_i}=\cdots=J_{n_{i+1}-1}=f_iR[X]$. Moreover $(f_i)_P=(f_{i+1})_P$ for any prime P of R such that either $e_{i+1}\in P$ or $e_i\notin P$; and if $e_{i+1}\notin P$ and $e_i\in P$, then $(f_i)_P=0$ and the image of f_{i+1} in $R_P[X]$ has degree n_{i+1}. The primes P of R such that $e_{i+1}\notin P$ are precisely those not containing $e_{i+1}-e_i$, and since $e_{i+1}-e_i$ is a nonzero idempotent, there do exist such primes. It follows that $\deg f_{i+1}=n_{i+1}$, the leading coefficient of f_{i+1} is $e_{i+1}-e_i$, and $(e_if_{i+1})=(f_i)$, for each nonnegative integer i. Also, it is easy to see, and shown for quasi-monics in [**M**, p. 167], that if f and g are almost quasi-monic polynomials generating the same ideal in $R[X]$, then $f=g$. Therefore, $e_if_{i+1}=f_i$ for each i. This completes the proof of the theorem in this direction.

For the converse, we first note as in [**BM**, Proposition 3] that if R has a strictly ascending sequence $(e_0)<(e_1)<(e_2)<\cdots$ of principal idempotent ideals, then $J=(e_0, e_1X, e_2X^2, \cdots)$ is such that J is not finitely generated and $R[X]/J$ is R-projective. We assume now the notation as in the statement of the theorem, viz. $\{e_i\}_{i=0}^{\infty}$ are idempotent elements of R such that $(e_0)<(e_1)<\cdots$, $e_0=f_0, f_1, \cdots$ are almost quasi-monic polynomials of $R[X]$ such that $e_if_{i+1}=f_i$ and f_{i+1} has leading coefficient

$e_{i+1}-e_i$, for each i. We note that $c(f_i)=e_iR$ for each i. We wish to show that $R[X]/I$ is R-projective. Let $\deg f_i=n_i$ and consider the R-submodule K of $R[X]$ generated by $(1-e_0), \cdots, (1-e_0)X^{n_1-1}$, $(1-e_1)X^{n_1}, \cdots, (1-e_1)X^{n_2-1}, \cdots,$. It is clear that K so defined is R-projective and is an R-module direct summand of $R[X]$. Let $\varphi: R[X]\to R[X]/I$ denote the natural homomorphism and let $\varphi(X)=\xi$. We show that φ restricted to K defines an isomorphism of K with $R[\xi]$. To show that $\varphi(K)=R[\xi]$ we check locally at primes P of R. If $c(I)=\bigcup_{i=0}^{\infty} e_iR\subseteq P$, then $(1-e_i)R_P=R_P$, so $\xi^iR_P\subseteq(\varphi(K))_P$ for each i. If $c(I)\not\subseteq P$, choose i minimal such that $e_i\notin P$. Then $f_iR_P[X]\neq 0$, so $1, \xi, \cdots, \xi^{n_i-1}$ generate $R_P[\xi]$; and $e_j\in P$, for $j<i$ implies that $(1-e_j)R_P=R_P$, so $\xi^kR_P\subseteq(\varphi(K))_P$ for $k<n_i$. We conclude that $\varphi(K)=R[\xi]$. It remains to show that $I\cap K=0$. Suppose $g=a_mX^m+\cdots+a_0\in I$ with $a_m\neq 0$. Then $g\in f_jR[X]$ for some j and $c(g)\subseteq e_jR$. Let i be chosen minimal such that $c(g)\subseteq e_iR$. From the fact that $e_ig=g$, we see that $g\in f_iR[X]$. Moreover, if $i>0$, then f_i is a monic polynomial in $(e_i-e_{i-1})R[X]$ of degree n_i, and $(e_i-e_{i-1})\neq 0$. Hence $\deg g=m\geqq n_i$. But if $g\in K$, then $a_mX^m\in K$, and $m\geqq n_i$ implies $a_m\in(1-e_i)R$. This would imply that $a_m\in e_iR\cap(1-e_i)R=0$. Therefore, $I\cap K=0$, which completes the proof of the theorem. Q.E.D.

We note the following Corollary.

COROLLARY. *If $R[X]/I$ is R-projective, then I is a projective ideal of $R[X]$.*

PROOF. If I is finitely generated, then from $R[X]/I$ R-flat it follows that I is a projective ideal [**OR**, Theorem 4.3, p. 402]. If I is not finitely generated, then, the notation being that of the theorem, $I=e_0I\oplus(e_1-e_0)I\oplus(e_2-e_1)I\oplus\cdots$ as $R[X]$-modules. Moreover, for each positive integer n, $(e_n-e_{n-1})I=(f_n-f_{n-1})R[X]$, which we observe is projective; for f_n-f_{n-1} is monic in $(e_n-e_{n-1})R[X]$ and therefore the annihilator in $R[X]$ of (f_n-f_{n-1}) is easily seen to be $[1-(e_n-e_{n-1})]R[X]$. Since $e_0I=e_0R[X]$ is also $R[X]$-projective, so is I.

REMARK 1. We note that the Corollary is no longer valid if "projective" is replaced by "flat". To see this, let R be an absolutely flat ring which is not noetherian. Then the projective global dimension of $R[X]$ is greater than one and so $R[X]$ contains an ideal I which is not projective [**R**, Theorem 9.8, p. 175]. But $R[X]/I$ is R-flat. More concretely, let A be an ideal of R which is not finitely generated. Then the ideal (A, X) of $R[X]$ is not projective, for by [**V**, Proposition 1.1, p. 270], if it were projective, it would be contained in a minimal prime of $R[X]$ and consequently consist entirely of zero-divisors.

REMARK 2. Let $\mathscr{S}$ denote the class of commutative rings R for which from $R[X]/I$ R-projective it follows that I is finitely generated. By the theorem, $\mathcal{S}$ consists precisely of those rings R which have only a finite number of idempotents. Thus, let R be a subring of the ring T. If $T \in \mathcal{S}$, then $R \in \mathcal{S}$. Of course, if $R \in \mathcal{S}$, T need not belong to $\mathcal{S}$. But suppose that R is a *retract* of T, that is, $T = R \oplus M$ as R-modules with M an ideal of T, and assume that $\bigcap_{n=0}^{\infty} M^n = 0$. An easy calculation shows that each idempotent of T belongs to R. Therefore, if $R \in \mathcal{S}$, $T \in \mathcal{S}$. In particular, this holds if T is a polynomial ring or a power series ring over R. It follows that R, $R[X]$ and $R[[X]]$ are simultaneously members of $\mathcal{S}$.

Since idempotents can easily be lifted modulo the nil-radical N of R [**L**, p. 73], $R \in \mathcal{S}$ if and only if $R/N \in \mathcal{S}$.

We conclude by observing that $\mathcal{S}$ is not closed under finite ring extension even inside the total quotient ring. For example, there exists a 1-dimensional reduced quasi-local ring R having an infinite number of minimal primes and having the property that the maximal ideal of R is the radical of a principal ideal (s) of R, cf. [**HO**, p. 5]. Since R is reduced, s is a regular element of R and $T = R[1/s]$ is a nonnoetherian absolutely flat ring. Hence $T \notin \mathcal{S}$.

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