PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, March 1974

IDEALS I OF $R[X]$ FOR WHICH $R[X]/I$ IS R-PROJECTIVE

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ABSTRACT. A characterization is given of those ideals I of the polynomial ring $R[X]$ such that $R[X]/I$ is R-projective. It is also shown that a commutative ring R has the property "$R[X]/I$ R-projective implies I is a finitely generated ideal" if and only if R has only a finite number of idempotents.

Let R be a commutative unitary ring and let I be an ideal of the polynomial ring $R[X]$. What conditions on I are necessary and sufficient in order that $R[X]/I$ be a projective R-module? Which rings R have the property that from $R[X]/I$ R-projective it follows that I is a finitely generated ideal? These two questions are raised in **[BM]** and our purpose here is to settle them. In **[BM]** it is shown that when $R[X]/I$ is projective, I is finitely generated if and only if I is principal. In this case, $I=fR[X]$ where f is *almost quasi-monic*, meaning that there exist pairwise orthogonal idempotents $e_0, e_1, \cdots, e_n$ of R such that $e_i f$ is a monic polynomial of $e_iR[X]$ for $0 \leqq i \leqq n$ and if $e=\sum_{i=0}^n e_i$, then $(1-e)f=0$. Therefore, the problem arises when I is not finitely generated and it is this case that our theorem treats.

A fact, which we shall use implicitly and which is recorded explicitly in **[BM]**, is that, for $R[X]/I$ R-projective and for P a prime ideal of R, $I_P=IR_P[X]$ is 0 or principal generated by a monic polynomial.

Another such fact is the following: Let J be an ideal of $R[X]$. Denote by $c(J)$ the ideal of R generated by those elements of R occurring as the coefficient of some element of J. If $R[X]/J$ is R-flat, then $c(J)$ is a *pure* ideal of R, that is, for each prime ideal P of R, $(c(J))_P=0$ or $(c(J))_P=R_P$ **[OR**, Corollary 1.3, p. 380]. Moreover, if $R[X]/J$ is R-projective, J is finitely generated if and only if $c(J)$ is finitely generated **[BM**, Theorem 3].

THEOREM. *The ring R has the property "$R[X]/I$ is R-projective implies I is finitely generated" if and only if R has only a finite number of*

Received by the editors June 28, 1973.

AMS (MOS) subject classifications (1970). Primary 13A15, 13B25, 13C10.

Key words and phrases. Polynomial ring, projective module, projective ideal, content, finitely generated ideal, idempotent element.

[1] During the writing of this paper the authors received partial support from the National Science Foundation.

idempotents. Moreover, for an ideal I of $R[X]$ which is not finitely generated, $R[X]/I$ is R-projective if and only if there exists an infinite set $\{e_i\}_{i=0}^{\infty}$ of idempotents in R such that $(e_0)<(e_1)<\cdots$, and a set $e_0=f_0, f_1, \cdots$ of almost quasi-monic polynomials in I such that $I=(f_0, f_1, \cdots)$, $e_i f_{i+1}=f_i$, and the leading coefficient of f_{i+1} is $e_{i+1}-e_i$, for each i.

PROOF. Suppose that $R[X]/I$ is R-projective and I is not finitely generated. Consider the R-submodules of $R[X]$ defined by $F_n=R+RX+\cdots+RX^n$ and $I_n=I\cap F_n$ for each nonnegative integer n. We first show that F_n/I_n and hence I_n are finitely generated projective R-modules. Consider the exact sequence

$$0\to F_n/I_n\to R[X]/I\to C_n\to 0$$

where the maps are the natural ones and C_n denotes the cokernel. Since F_n/I_n is a finitely generated R-module and since $R[X]/I$ is a countably generated projective R-module, to prove that F_n/I_n is R-projective, it suffices, by virtue of [**J**, Lemma 2, p. 57], to prove that C_n is R-flat. For this, we look locally at primes of R. Thus, let P be a prime ideal of R. If $P\supseteq c(I)$, then $I_P=(I_n)_P=0$ and so $(F_n)_P/(I_n)_P\cong R_P+R_PX+\cdots+R_PX^n$ and $R_P[X]/I_P\cong R_P[X]$. It follows that $(C_n)_P\cong R_PX^{n+1}+R_PX^{n+2}+\cdots$ is a free R_P-module. If $P\not\supseteq c(I)$, then $I_P\neq 0$, and since $R[X]/I$ is R-projective, I_P is generated by a monic polynomial of nonnegative degree in $R_P[X]$. Thus, there exists a polynomial $f\in I$ such that the image of f in $R_P[X]$ generates I_P. By multiplying, if necessary, by an appropriate element of $R\backslash P$, we may assume that f is such that the leading coefficient of f is a unit in R_P. Under this assumption, if $s=\text{degree}(f)>n$, then $(I_n)_P=0$. Thus, $(F_n)_P/(I_n)_P\cong R_P+R_PX+\cdots+R_PX^n$, $R_P[X]/I_P\cong R_P+R_PX+\cdots+R_PX^s$, and $(C_n)_P$ is R_P-free. If $s\leqq n$, then our assumption tells us that $f\in I_n$, and in fact, $f, Xf, \cdots, X^{n-s}f\in I_n$. Moreover, $1, X, \cdots, X^{s-1}, f, Xf, \cdots, X^{n-s}f$ form a free basis for $(F_n)_P$ as an R_P-module. Since any polynomial in $(I_n)_P$ has degree $\geqq s$, it follows that $f, Xf, \cdots, X^{n-s}f$ generate $(I_n)_P$. Thus $(F_n)_P/(I_n)_P$ is a free R_P-module generated by the images of $1, X, \cdots, X^{s-1}$, $(F_n)_P/(I_n)_P\cong R_P[X]/I_P$, and in this case $(C_n)_P=0$. We conclude that F_n/I_n is R-projective. Therefore, the sequence of R-modules $0\to I_n\to F_n\to F_n/I_n\to 0$ splits and I_n is a finitely generated projective R-module. It follows, for example from [**OR**, Corollary 1.3, p. 380], that $c(I_n)$=ideal of R generated by the coefficients of elements of I of degree $\leqq n$ is principal generated by an idempotent. Let $c(I_n)=eR$, where e is an idempotent element of R, and let $J=I_nR[X]$.

CLAIM. *$R[X]/J$ is R-isomorphic to $(1-e)R[X]\oplus eF_n/I_n$ and therefore $R[X]/J$ is R-projective.*

PROOF OF CLAIM. We have $c(J)=c(I_n)=eR$, so $R[X]/J=(1-e)R[X]\oplus eR[X]/J$. Thus, we need only show that eF_n/I_n is R-isomorphic to $eR[X]/J$. Since $I_n=I\cap F_n$, $J\cap F_n=I_n$ and F_n/I_n is canonically embedded in $R[X]/J$. Thus, eF_n/I_n is a submodule of $eR[X]/J$ and if $e, e\theta, \cdots, e\theta^n$ denote the images of $e, eX, \cdots, eX^n$ in $eR[X]/J$, then these elements are in the submodule eF_n/I_n. Hence, to show $eF_n/I_n=eR[X]/J$, it will suffice to show that $e, e\theta, \cdots, e\theta^n$ generate the R-module $eR[X]/J$. We show this by showing $e, e\theta, \cdots, e\theta^n$ generate $eR[X]/J$ locally at any prime P of R. If $e\in P$, then $(eR[X]/J)_P=0$; and if $e\notin P$, then $c(I_n)\nsubseteq P$, so $J_P=I_P$ is generated by a monic polynomial in $R_P[X]$ of degree $\leqq n$. Thus, the images of $e, e\theta, \cdots, e\theta^n$ in $(eR_P[X]/J)$ generate $(eR_P[X]/J)_P$ as an R_P-module. We conclude that $eF_n/I_n=eR[X]/J$, which completes the proof of the claim.

We clearly have $\bigcup_{n=0}^{\infty} c(I_n)=c(I)$ and by assumption I, and therefore $c(I)$, is not finitely generated. Hence, the ascending sequence of ideals $c(I_0)\subseteqq c(I_1)\subseteqq\cdots$ does not become stationary and we may assume that the notation is as follows: $e_0R=c(I_0)=\cdots=c(I_{n_1-1})<e_1R=c(I_{n_1})=\cdots=c(I_{n_2-1})<e_2R=c(I_{n_2})=\cdots$, the e_i's idempotents of R and the n_i's a strictly ascending sequence of positive integers. Let $J_k=I_kR[X]$. Then J_k is a finitely generated ideal of $R[X]$ and we have shown above that $R[X]/J_k$ is R-projective. Hence by [**BM**, Theorem 3], J_k is a principal ideal of $R[X]$ generated by an almost quasi-monic polynomial. We observe that if $m\leqq n$ and $c(J_m)=c(J_n)$, then $J_m=J_n$. For if P is a prime of R and $c(J_m)=c(J_n)\subseteqq P$, then $(J_m)_P=(J_n)_P=0$; and if $c(J_m)\nsubseteq P$, then $(J_m)_P=I_P\supseteqq(J_n)_P$, so $(J_m)_P=(J_n)_P$. We denote the almost quasi-monic generator of J_{n_i} by f_i. Note that $c(f_i)=e_i$ and $J_{n_i}=\cdots=J_{n_{i+1}-1}=f_iR[X]$. Moreover $(f_i)_P=(f_{i+1})_P$ for any prime P of R such that either $e_{i+1}\in P$ or $e_i\notin P$; and if $e_{i+1}\notin P$ and $e_i\in P$, then $(f_i)_P=0$ and the image of f_{i+1} in $R_P[X]$ has degree n_{i+1}. The primes P of R such that $e_{i+1}\notin P$ are precisely those not containing $e_{i+1}-e_i$, and since $e_{i+1}-e_i$ is a nonzero idempotent, there do exist such primes. It follows that $\deg f_{i+1}=n_{i+1}$, the leading coefficient of f_{i+1} is $e_{i+1}-e_i$, and $(e_if_{i+1})=(f_i)$, for each nonnegative integer i. Also, it is easy to see, and shown for quasi-monics in [**M**, p. 167], that if f and g are almost quasi-monic polynomials generating the same ideal in $R[X]$, then $f=g$. Therefore, $e_if_{i+1}=f_i$ for each i. This completes the proof of the theorem in this direction.

For the converse, we first note as in [**BM**, Proposition 3] that if R has a strictly ascending sequence $(e_0)<(e_1)<(e_2)<\cdots$ of principal idempotent ideals, then $J=(e_0, e_1X, e_2X^2, \cdots)$ is such that J is not finitely generated and $R[X]/J$ is R-projective. We assume now the notation as in the statement of the theorem, viz. $\{e_i\}_{i=0}^{\infty}$ are idempotent elements of R such that $(e_0)<(e_1)<\cdots$, $e_0=f_0, f_1, \cdots$ are almost quasi-monic polynomials of $R[X]$ such that $e_if_{i+1}=f_i$ and f_{i+1} has leading coefficient

$e_{i+1}-e_i$, for each i. We note that $c(f_i)=e_iR$ for each i. We wish to show that $R[X]/I$ is R-projective. Let $\deg f_i=n_i$ and consider the R-submodule K of $R[X]$ generated by $(1-e_0), \cdots, (1-e_0)X^{n_1-1}$, $(1-e_1)X^{n_1}, \cdots, (1-e_1)X^{n_2-1}, \cdots,$. It is clear that K so defined is R-projective and is an R-module direct summand of $R[X]$. Let $\varphi: R[X]\to R[X]/I$ denote the natural homomorphism and let $\varphi(X)=\xi$. We show that φ restricted to K defines an isomorphism of K with $R[\xi]$. To show that $\varphi(K)=R[\xi]$ we check locally at primes P of R. If $c(I)=\bigcup_{i=0}^{\infty} e_iR\subseteq P$, then $(1-e_i)R_P=R_P$, so $\xi^i R_P\subseteq(\varphi(K))_P$ for each i. If $c(I)\not\subseteq P$, choose i minimal such that $e_i\notin P$. Then $f_iR_P[X]\neq 0$, so $1, \xi, \cdots, \xi^{n_i-1}$ generate $R_P[\xi]$; and $e_j\in P$, for $j<i$ implies that $(1-e_j)R_P=R_P$, so $\xi^k R_P\subseteq(\varphi(K))_P$ for $k<n_i$. We conclude that $\varphi(K)=R[\xi]$. It remains to show that $I\cap K=0$. Suppose $g=a_mX^m+\cdots+a_0\in I$ with $a_m\neq 0$. Then $g\in f_jR[X]$ for some j and $c(g)\subseteq e_jR$. Let i be chosen minimal such that $c(g)\subseteq e_iR$. From the fact that $e_ig=g$, we see that $g\in f_iR[X]$. Moreover, if $i>0$, then f_i is a monic polynomial in $(e_i-e_{i-1})R[X]$ of degree n_i, and $(e_i-e_{i-1})\neq 0$. Hence $\deg g=m\geqq n_i$. But if $g\in K$, then $a_mX^m\in K$, and $m\geqq n_i$ implies $a_m\in(1-e_i)R$. This would imply that $a_m\in e_iR\cap(1-e_i)R=0$. Therefore, $I\cap K=0$, which completes the proof of the theorem. Q.E.D.

We note the following Corollary.

COROLLARY. *If $R[X]/I$ is R-projective, then I is a projective ideal of $R[X]$.*

PROOF. If I is finitely generated, then from $R[X]/I$ R-flat it follows that I is a projective ideal [**OR**, Theorem 4.3, p. 402]. If I is not finitely generated, then, the notation being that of the theorem, $I=e_0I\oplus(e_1-e_0)I\oplus(e_2-e_1)I\oplus\cdots$ as $R[X]$-modules. Moreover, for each positive integer n, $(e_n-e_{n-1})I=(f_n-f_{n-1})R[X]$, which we observe is projective; for f_n-f_{n-1} is monic in $(e_n-e_{n-1})R[X]$ and therefore the annihilator in $R[X]$ of (f_n-f_{n-1}) is easily seen to be $[1-(e_n-e_{n-1})]R[X]$. Since $e_0I=e_0R[X]$ is also $R[X]$-projective, so is I.

REMARK 1. We note that the Corollary is no longer valid if "projective" is replaced by "flat". To see this, let R be an absolutely flat ring which is not noetherian. Then the projective global dimension of $R[X]$ is greater than one and so $R[X]$ contains an ideal I which is not projective [**R**, Theorem 9.8, p. 175]. But $R[X]/I$ is R-flat. More concretely, let A be an ideal of R which is not finitely generated. Then the ideal (A, X) of $R[X]$ is not projective, for by [**V**, Proposition 1.1, p. 270], if it were projective, it would be contained in a minimal prime of $R[X]$ and consequently consist entirely of zero-divisors.

REMARK 2. Let $\mathcal{S}$ denote the class of commutative rings R for which from $R[X]/I$ R-projective it follows that I is finitely generated. By the theorem, $\mathcal{S}$ consists precisely of those rings R which have only a finite number of idempotents. Thus, let R be a subring of the ring T. If $T \in \mathcal{S}$, then $R \in \mathcal{S}$. Of course, if $R \in \mathcal{S}$, T need not belong to $\mathcal{S}$. But suppose that R is a *retract* of T, that is, $T=R\oplus M$ as R-modules with M an ideal of T, and assume that $\bigcap_{n=0}^{\infty} M^n=0$. An easy calculation shows that each idempotent of T belongs to R. Therefore, if $R \in \mathcal{S}$, $T \in \mathcal{S}$. In particular, this holds if T is a polynomial ring or a power series ring over R. It follows that R, $R[X]$ and $R[[X]]$ are simultaneously members of $\mathcal{S}$.

Since idempotents can easily be lifted modulo the nil-radical N of R [**L**, p. 73], $R \in \mathcal{S}$ if and only if $R/N \in \mathcal{S}$.

We conclude by observing that $\mathcal{S}$ is not closed under finite ring extension even inside the total quotient ring. For example, there exists a 1-dimensional reduced quasi-local ring R having an infinite number of minimal primes and having the property that the maximal ideal of R is the radical of a principal ideal (s) of R, cf. [**HO**, p. 5]. Since R is reduced, s is a regular element of R and $T=R[1/s]$ is a nonnoetherian absolutely flat ring. Hence $T \notin \mathcal{S}$.

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