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THE CONDITION $\text{Ext}^i(M, R)=0$ FOR MODULES OVER LOCAL ARTIN ALGEBRAS $(R, \mathfrak{M})$ WITH $\mathfrak{M}^2=0$

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ABSTRACT. Let M be a finitely generated module over a (not necessarily commutative) local Artin algebra $(R, \mathfrak{M})$ with $\mathfrak{M}^2=0$. It is known that when R is Gorenstein (i.e. of finite injective dimension) $M=\Sigma R\oplus\Sigma R/\mathfrak{M}$. For R not Gorenstein we describe all M with $\text{Ext}^1(M, R)=0$ and show that $\text{Ext}^i(M, R)=0$ for some $i>1$ if and only if M is free. It follows that for R not Gorenstein all reflexives are free. We also calculate the lengths of all the $\text{Ext}^i(M, R)$. As an application we show that if $(R, \mathfrak{M})$ is a commutative Cohen-Macaulay local ring of dimension d which is not Gorenstein, if $R/\mathfrak{M}^2$ is Artin and $(x_1, \cdots, x_d)$ is a system of parameters with $\mathfrak{M}^2$ contained in the ideal $(x_1, \cdots, x_d)$ and if M is a finitely generated R-module with $\text{Ext}^i(M, R)=0$ for $1\leqq i\leqq 2d+2$, then M is free.

We call $(R, \mathfrak{M})$ a local Artin algebra if $R/\mathfrak{M}$ is a division ring, where $\mathfrak{M}$ is the Jacobson radical of R, if the center of R is an Artin ring, and if R if a finitely generated module over its center. We say R is Gorenstein if it is of finite injective dimension as an R-module. Throughout this paper all modules will be finitely generated.

Let $(R, \mathfrak{M})$ be a local Artin algebra. It is well known that every finitely generated left R-module M has a projective cover (i.e. an epimorphism $\varphi: P\to M$ minimal in the sense that Ker $\varphi\subseteq\mathfrak{M}P$) which is unique up to isomorphism [3], and that M has no projective (free) direct summands if and only if $M^*=\text{Hom}_R(M, R)$ is isomorphic to $\text{Hom}_R(M, \mathfrak{M})$ [2]. Further, for each finitely generated left R-module M there is a minimal presentation—i.e. an exact sequence $F_1\to F_0\to M\to 0$ with $\varphi_0: F_0\to M$ and $\varphi_1: F_1\to \text{Ker}\ \varphi_0$ projective covers, F_0 and F_1 free and finitely generated. We use this (unique) minimal presentation to define the parameters g_M and r_M.

DEFINITION. Let M be a finitely generated left module over a local Artin algebra $(R, \mathfrak{M})$. Let

$$g_M = \ell(F_0/\mathfrak{M}F_0) \quad \text{the number of generators of } M,$$
$$r_M = \ell(F_1/\mathfrak{M}F_1) \quad \text{the number of relations of } M,$$

where $\ell(\ )$ means the length as a left R-module.

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We remark that if $(R, \mathfrak{M})$ is a local Artin algebra and M is a 2-sided R-module then the lengths of M as a left and as a right R-module coincide.

PROPOSITION 1. *Let $(R, \mathfrak{M})$ be a local Artin algebra with $\mathfrak{M}^2=0$. Let $n=\ell(\mathfrak{M})\geqq 1$ and let M be a finitely generated left R-module with no free direct summands. Then*:

(a) $\ell(M)=(n+1)g_M-r_M$,
(b) $\ell(M^*)=ng_M$,
(c) $\ell(\mathrm{Ext}^1(M, R))=nr_M-g_M$,
(d) $\ell(\mathrm{Ext}^i(M, R))=n^{i-2}r_M(n^2-1)$, $i\geqq 2$.

PROOF. We write g for g_M and r for r_M. Let $0\to K\to R^g\to M\to 0$ be exact with $R^g\to M\to 0$ a projective cover. As $K\subseteq \mathfrak{M}R^g$, $\ell(M/\mathfrak{M}M)=g_M=g$. Also, as $\mathfrak{M}^2=0$, $\mathfrak{M}K=0$ and K is a direct sum of copies of $R/\mathfrak{M}$. $0\to\mathfrak{M}\to R\to R/\mathfrak{M}\to 0$ is the projective cover of $R/\mathfrak{M}$, and $\mathfrak{M}$ is isomorphic to n copies of $R/\mathfrak{M}$ (direct sum). Since $R^r\to K\to 0$ is a projective cover, $g_K=\ell(K)=r_M=r$, and by the same reasoning $r_K=\ell(\mathfrak{M}R^r)=nr$ $(=nr_M)$.

(a) From $\ell(R)=\ell(\mathfrak{M})+1=n+1$, we have

$$\ell(M)=\ell(R^g)-\ell(K)=(n+1)g-r.$$

(b) As M is without free summands and $\mathfrak{M}^2=0$, $M^*\approx\mathrm{Hom}_R(M, \mathfrak{M})\approx \mathrm{Hom}_R(M/\mathfrak{M}M, \mathfrak{M})\approx\mathrm{Hom}_{R/\mathfrak{M}}(M/\mathfrak{M}M, \mathfrak{M})$.

As the lengths of $R/\mathfrak{M}$ modules as R-modules and as $R/\mathfrak{M}$-modules coincide, this last has length $\ell(M/\mathfrak{M}M)\cdot\ell(\mathfrak{M})=ng$.

(c) From the exact sequence $0\to M^*\to(R^g)^*\to K^*\to\mathrm{Ext}^1(M, R)\to 0$, we have

$$\begin{aligned}\ell(\mathrm{Ext}^1(M, R)) &= \ell(M^*)+\ell(K^*)-\ell(R^{g*})\\ &= ng_M+ng_K-(n+1)g_M\\ &= ng_K-g_M=nr_M-g_M=nr-g.\end{aligned}$$

(d) $\mathrm{Ext}^2(M, R)\approx\mathrm{Ext}^1(K, R)$ is of length $nr_K-g_K=n(nr)-r=r(n^2-1)=r_M(n^2-1)$. By induction on i, $\mathrm{Ext}^{i+1}(M, R)=\mathrm{Ext}^i(K, R)$ is of length $n^{i-2}r_K(n^2-1)=n^{i-1}r(n^2-1)=n^{i-1}r_M(n^2-1)$ for $i\geqq 2$.

PROPOSITION 2. *Let $(R, \mathfrak{M})$ be a local Artin algebra with $\mathfrak{M}^2=0$, and let M be a (finitely generated) left R-module. If $n=\ell(\mathfrak{M})\geqq 2$ then the following are equivalent*:

(a) *M is free.*
(b) $\mathrm{Ext}^1(M, R)=\mathrm{Ext}^2(M, R)=0$.
(c) $\mathrm{Ext}^i(M, R)=0$ *for some* $i\geqq 2$.

PROOF. Trivially (a)$\Rightarrow$(b)$\Rightarrow$(c). If $M\neq(0)$ is a module without free summands, $n\geqq 2$, and some $\mathrm{Ext}^i(M, R)=0$, then by the last formula in Proposition 1, $r_M=0$. Hence $K=0$ and M is free.

We obtain easily the well-known result:

COROLLARY 3. *Let* $(R, \mathfrak{M})$ *be a local Artin algebra with* $\mathfrak{M}^2=0$. *Let* $n=\ell(\mathfrak{M})$. *Then R is Gorenstein if and only if* $n=0$ *or* 1. *If R is Gorenstein, then* Injective dimension$_R$ $R=0$ (*i.e., R is injective*) *and all R-modules are of the form* $\Sigma R \oplus \Sigma R/\mathfrak{M}$.

PROOF. When $n=0$, R is a division ring and the statement is true trivially. If R is Gorenstein, $\mathrm{Ext}^i(R/\mathfrak{M}, R)=0$ for some $i \geqq 1$ and also for some $i \geqq 2$. Since $r_{R/\mathfrak{M}}=n$, from Proposition 1, we obtain $n^2-1=0$ and $n=1$. If $n=1$, $\mathrm{Ext}^2(M, R)=0$ for all finitely generated M by Proposition 1 and R is of finite injective dimension. In fact, since $r_{R/\mathfrak{M}}=n=1$ and $g_{R/\mathfrak{M}}=1$, we find $\mathrm{Ext}^1(R/\mathfrak{M}, R)=0$ and hence

$$\text{Injective dimension}_R\, R=0.$$

If R is Gorenstein, and $M \neq R$ is indecomposable, then M is torsion-free (for example, by the sequence $S(\ )$ applied to R, below). So $M \subseteq P$ for some P projective and as M is without free summands $M \subseteq \mathfrak{M}P$ [**2**]. Since $\mathfrak{M}^2=0$, M is not faithful. Hence, $\mathfrak{M}$ is the annihilator of M and $M=R/\mathfrak{M}$. R and $R/\mathfrak{M}$ are the only indecomposable R-modules.

To investigate reflexive modules we introduce the map T. If R is a local Artin algebra and M is a finitely generated left R-module with minimal presentation

$$F_1 \to F_0 \to M \to 0 \tag{1}$$

(F_0, F_1 finitely generated and free), then we define $T(M)=\mathrm{Coker}(F_0^* \to F_1^*)$ so that

$$0 \to M^* \to F_0^* \to F_1^* \to T(M) \to 0 \tag{2}$$

is exact. When M has no free (direct) summands and (1) is a minimal presentation of M, then (2) is a minimal presentation of $T(M)$, and (since $T(M)$ is without free summands) this presentation of $T(M)$ produces $T^2(M)$ with $T^2(M) \approx M$. (To show (2) is a minimal presentation, write down the definitions of the maps. $\pi : F_1^* \to T(M)$ is minimal when M is without free summands so that $M^*=\mathrm{Hom}_R(M, \mathfrak{M})$, since $\varphi_0 : F_0 \to M$ is minimal. $\varphi_1^* : F_0 \to \mathrm{Ker}\, \pi$ is minimal since $\varphi_1 : F_1 \to \mathrm{Ker}\, \varphi_0$ is minimal.) Also, as minimal presentations of M_1 and M_2 induce a minimal presentation of $M_1 \oplus M_2$, $T(M_1 \oplus M_2) \approx T(M_1) \oplus T(M_2)$.

One of the most important facts about T is that if M is a left R-module then

$$S(\ ) \qquad \begin{aligned} 0 \to \mathrm{Ext}^1(T(M), \_\_) \to \_\_ \otimes M \to \mathrm{Hom}(M^*, \_\_) \\ \to \mathrm{Ext}^2(T(M), \_\_) \to 0 \end{aligned}$$

is an exact sequence of functors of right R-modules [**1**, Proposition 6.3], or [**4**, p. 43].

In particular, when we examine the sequence $S(R)$ for the right R-module R, the second map is the canonical $M \to M^{**}$ and we see that M is reflexive ($M \approx M^{**}$) if and only if $\mathrm{Ext}^1(T(M), R) = \mathrm{Ext}^2(T(M), R) = 0$.

PROPOSITION 4. *Let $(R, \mathfrak{M})$ be a local Artin algebra with $\mathfrak{M}^2 = 0$. If $n = \ell(\mathfrak{M}) \geqq 2$ then any reflexive module is free. If $n = \ell(\mathfrak{M}) < 2$ then every module is reflexive.*

PROOF. We use the (unique) minimal presentation of M to define $T(M)$. If $n = \ell(\mathfrak{M}) < 2$, R is Gorenstein, hence all $\mathrm{Ext}^1(T(M), R) = 0 = \mathrm{Ext}^2(T(M), R)$ and every M is reflexive. If $n \geqq 2$, let M be a reflexive module without free summands. Then $\mathrm{Ext}^2(T(M), R) = 0$ and by Proposition 2, $T(M)$ is free. But then $0 \to T(M) \to T(M) \to 0$ is the unique minimal presentation of $T(M)$, so $M = T(T(M)) = 0$. Thus the only reflexives are free.

PROPOSITION 5. *Let $(R, \mathfrak{M})$ be a local Artin algebra with $\mathfrak{M}^2 = 0$. Then the following conditions are equivalent*:

(a) $\mathrm{Ext}^1(M, R) = 0$.

(b) *M is a direct sum $\Sigma R \oplus \Sigma T(R/\mathfrak{M})$.*

(c) *If $0 \to A \to B \to M \to 0$ and $0 \to A/\mathfrak{M}A \to B/\mathfrak{M}B \to M/\mathfrak{M}M \to 0$ are both exact, then the first sequence splits.*

(d) *If $0 \to A \to B \to M \to 0$ is exact and minimal presentations of A and M induce a minimal presentation of B, then the sequence splits.*

PROOF. (c) and (d) are equivalent since the condition that minimal presentations of A and M induce a minimal presentation of B is exactly the condition that $g_A + g_M = g_B$. Let $n = \ell(\mathfrak{M})$. Now if $M = T(R/\mathfrak{M})$ then $r_M = g_{R/\mathfrak{M}} = 1$ and $g_M = r_{R/\mathfrak{M}} = n$, so $\ell(\mathrm{Ext}^1(M, R)) = nr_M - g_M = 0$. Hence (b)$\Rightarrow$(a).

To show (a)$\Rightarrow$(b), from the sequence before Proposition 4 we have the exact sequence $0 \to \mathrm{Ext}^1(T^2(M), R) \to T(M) \to T(M)^{**}$. If $M = F \oplus M'$, M' without free summands, then $T^2(M) = T^2(M') = M'$. By our assumption $0 = \mathrm{Ext}^1(T^2(M), R) = 0$ and $0 \to T(M) \to T(M)^{**}$ is exact—i.e. $T(M)$ is torsionless. Thus $T(M) \subseteq F'$ for some free F'. As also $T(M)$ is without projective summands, $T(M) \subseteq F'\mathfrak{M}$. As $\mathfrak{M}^2 = 0$, $T(M)$ is a direct sum of copies of $R/\mathfrak{M}$ and therefore $M' \approx T^2(M) = \Sigma T(R/\mathfrak{M})$.

To show (c)$\Rightarrow$(a) consider a sequence $0 \to R \to N \to M \to 0$. Now R is not contained in $\mathfrak{M}N$, else $\mathfrak{M} \subseteq \mathfrak{M}^2 N = 0$. As $\ell_R(R/\mathfrak{M}) = 1$, $R/\mathfrak{M} \subseteq N/\mathfrak{M}N$ and $0 \to R/\mathfrak{M}R \to N/\mathfrak{M}N \to M/\mathfrak{M}M \to 0$ is exact. By our assumption (c) the original sequence is split and $\mathrm{Ext}^1(M, R) = 0$.

Finally we show (a)$\Rightarrow$(d). If minimal presentations of A and M induce a minimal presentation of B, then dualizing the presentations $G_X \to F_X \to X \to 0$, $X=A, B, M$, we obtain the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M^* & \longrightarrow & F_M^* & \longrightarrow & G_M^* & \longrightarrow & T(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \text{I} & \downarrow & & & & \\
0 & \longrightarrow & B^* & \longrightarrow & F_B^* & \longrightarrow & G_B^* & \longrightarrow & T(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & \text{II} & \downarrow & & & & \\
0 & \longrightarrow & A^* & \longrightarrow & F_A^* & \longrightarrow & G_A^* & \longrightarrow & T(A) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

in which all rows are exact, and the second and third columns are split exact in such a way as to make boxes I and II commutative. By the snake lemma, the sequence $0 \to M^* \to B^* \to A^* \to T(M) \to T(B) \to T(A) \to 0$ is exact. But $\mathrm{Coker}(B^* \to A^*) = \mathrm{Ext}^1(M, R) = 0$ by assumption. Hence $0 \to T(M) \to T(B) \to T(A) \to 0$ is exact, the maps being induced from the G-column. The splitting maps on the G-column induce maps $T(A) \to T(B)$ and $T(B) \to T(M)$. It is easy to show that, as the F and G-columns both split, the induced maps on the T-column split it, and hence $T(B) \approx T(M) \oplus T(A)$. Hence $B \approx T^2(B) \approx T(T(M) \oplus T(A)) \approx T^2(M) \oplus T^2(A) \approx M \oplus A$ and the original sequence splits.

COROLLARY 6. *Let* $(R, \mathfrak{M})$ *be a local Artin algebra with* $\mathfrak{M}^2=0$. *If* $0 \to A \to B \to M \to 0$ *is exact*, $\mathrm{Ext}^1(M, R)=0$, *and B is finitely generated without free direct summands, then the sequence splits.*

PROOF. We have $0 \to A \to B \to M \to 0$ and $0 \to M^* \to B^* \to A^* \to 0$ exact. If M had free direct summands, so would B. Likewise if A^* had free direct summands so would A^* and hence B^*—contrary to $B^* = \mathrm{Hom}_R(B, \mathfrak{M})$ (since B has no free summands). Thus the dual sequence gives $ng_B = \ell(B^*) = \ell(M^*) + \ell(A^*) = ng_M + ng_A$, since A, B, M are all without free summands. From the equivalence of (a) and (d) in Proposition 5 and $\mathrm{Ext}^1(M, R)=0$ it follows that the original sequence splits.

As an application of the above, we may generalize Proposition 2 to Cohen-Macaulay rings.

PROPOSITION 7. *Let* $(R, \mathfrak{M})$ *be a commutative Cohen-Macaulay local ring of dimension d which is not Gorenstein. Let M be a finitely generated* *left R-module. If* $(x_1, \cdots, x_d)$ *is a system of parameters such that* $\mathfrak{M}^2 \subseteq (x_1, \cdots, x_d) = \Sigma R x_i$ *and* $R/\mathfrak{M}^2$ *is Artin, then* $\operatorname{Ext}^i(M, R)=0$ *for* $1 \leqq i \leqq 2d+2$ *implies that M is free.*

PROOF. When $d=0$, $\mathfrak{M}^2=0$. Since R is not Gorenstein, $n=\ell(\mathfrak{M}) \geqq 2$ and the result is just Proposition 2. Assume the result for $0 \leqq d \leqq k-1$, and we show it for $d=k$. Let $x=x_1$. We consider a projective cover of $M: 0 \to M_1 \to F_0 \to M \to 0$, and show by the inductive hypothesis that M_1/M_1x is a free R/Rx-module. From $0 \to R \to R \to R/Rx \to 0$ exact, where the first map is multiplication by x, and the hypotheses on M, we conclude $\operatorname{Ext}^i_R(M_1, R/Rx)=0$, for $1 \leqq i \leqq 2k$. As $x, x_2, \cdots, x_d$ is a R-regular sequence, x is not a zero-divisor on F_0, or its submodule M_1. As x is M_1-regular there are isomorphisms of R/Rx-modules [**4**, p. 1]:

$$(\dagger) \qquad \begin{aligned} \operatorname{Ext}^i_R(M_1, \ ) &\approx \operatorname{Ext}^i_{R/Rx}(M_1/M_1x, \ ), \\ \operatorname{Ext}^i_R(\ , R) &\approx \operatorname{Ext}^{i+1}_{R/Rx}(\ , R/Rx) \quad \text{for } i \geqq 1. \end{aligned}$$

Hence $\operatorname{Ext}^i_{R/Rx}(M_1/M_1x, R/Rx)=0$ for $1 \leqq i \leqq 2k$. Also, R/Rx is a local Cohen-Macaulay ring of dimension $k-1$ with $x_2, \cdots, x_d$ a system of parameters satisfying the hypotheses of the theorem. Applying the second isomorphism of (†) to the R/Rx-module $R/\mathfrak{M}$, we see that R is Gorenstein (i.e. of finite injective dimension) if and only if R/Rx is. By the inductive hypotheses M_1/M_1x is R/Rx-free, and by the first isomorphism of (†) $0=\operatorname{Ext}^1_R(M_1, R/\mathfrak{M})$. Thus M_1 is R-free and M is of projective dimension 1 or 0. But for any exact sequence $0 \to K \to R^g \to N \to 0$ of R-modules, we have $\cdots \to \operatorname{Ext}^1(M, R^g) \to \operatorname{Ext}^1(M, N) \to \operatorname{Ext}^2(M, K)=0$. From $0= \operatorname{Ext}^1(M, R)$, we have $\operatorname{Ext}^1(M, N)=0$ for all finitely generated N and M is free.

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