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ON HUPPERT'S CONDITION B

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ABSTRACT. Let $\mathscr{F}$ be a saturated formation of finite soluble groups. $\mathscr{F}$ is said to satisfy condition B if and only if (a) $\mathscr{F}$ is subgroup-closed, and (b) $G \in \mathscr{F}$ and N a minimal normal subgroup of G implies $\mathrm{Aut}(N) \in \mathscr{F}$. The purpose of this note is to characterize those saturated formations of finite soluble groups which satisfy condition B.

Let $\mathscr{F}$ be a saturated formation of finite soluble groups. Then $\mathscr{F}$ is said to satisfy condition B (cf. Huppert [**6**, p. 569]) if and only if (a) $\mathscr{F}$ is subgroup-closed, and (b) $G \in \mathscr{F}$ and N a minimal normal subgroup of G implies $\mathrm{Aut}(N) \in \mathscr{F}$, where $\mathrm{Aut}(N)$ denotes the group of automorphisms of N. The purpose of this note is to characterize those saturated formations of finite soluble groups which satisfy condition B. Our characterization answers the question raised by Professor Klaus Doerk of Mainz.

In this paper only finite soluble groups are considered. $\mathscr{F}$ will always denote a saturated formation and $\{\mathscr{F}(p)\}$ the canonical definition of $\mathscr{F}$; that is $\{\mathscr{F}(p)\}$ defines $\mathscr{F}$ locally and $\mathscr{S}_p\mathscr{F}(p)=\mathscr{F}(p)\subseteq\mathscr{F}$ for each prime p. The set of primes π for which $\mathscr{F}(p)$ is nonempty is called the characteristic of $\mathscr{F}$. The concepts and notation are standard and we refer the reader to Carter and Hawkes [**1**], Gaschütz [**2**] and Huppert [**4**].

Let $r_p(G)$ denote the p-rank of G and $\bar{r}_p(G)$ the arithmetic p-rank of G. (For the definitions of the terms p-rank and arithmetic p-rank we refer the reader to §§5 and 8 of Chapter VI of [**4**].)

Let π be a nonempty set of prime numbers. Then π is said to be *special* if whenever $p \in \pi$, p odd, then the prime divisors of $p-1$ also belong to π. Further, π is said to be an *extra-special* set of primes if π is special and $3 \in \pi$ whenever $2 \in \pi$.

For any set of primes π, let $\mathscr{Y}_\pi$ denote the saturated formation of finite supersoluble groups G whose order $|G|$ is a π-number.

Let π be a set of extra-special prime numbers and let $\mathscr{X}_\pi$ denote the class of all finite soluble π-groups G with the properties:

(a) G', the derived subgroup of G, is nilpotent.

(b) $r_p(G)\leqq 1$ if $p\neq 2$ and $r_2(G)\leqq 2$.

One easily checks that $\mathscr{X}_\pi$ is a subgroup-closed formation. $\mathscr{X}_\pi$ is saturated

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For let $G/\phi(G) \in \mathscr{X}_\pi$, where $\phi(G)$ is the Frattini subgroup of G. Hence, G is a π-group and G' is nilpotent since $G'\phi(G)/\phi(G)$ is nilpotent. Assume that the prime 2 divides $|G|$. By Hilfssatz 5 of Huppert [5] it follows that $r_2(G) \leqq \bar{r}_2(G) \leqq 2$. Let p be an odd prime divisor of $|G|$. By Satz 8.4 of Huppert [4, p. 713] $r_p(G) \leqq \bar{r}_p(G) = \bar{r}_p(G/\phi(G)) = 1$. Therefore, $G \in \mathscr{X}_\pi$, and so $\mathscr{X}_\pi$ is a saturated formation. We also note that $\mathscr{X}_\pi$ satisfies condition B. For let $G \in \mathscr{X}_\pi$ and let N be a minimal normal subgroup of G whose order is a power of the prime p. If p is odd, then Aut $N \cong C_{p-1}$, the cyclic group of order $p-1$, which belongs to $\mathscr{X}_\pi$. If $p=2$, then we can assume $|N|=4$, whence Aut $N \cong S_3$ which belongs to $\mathscr{X}_\pi$.

Let π be an extra-special set of prime numbers and let $p \in \pi$. If p is odd, let $\mathscr{X}(p)$ denote the formation of Abelian groups of exponent dividing $p-1$. If $p=2$, let $\mathscr{X}(2)$ denote the formation generated by $S_3 \times C_3$, where C_3 is the cyclic group of order 3 and S_3 is the symmetric group on three symbols. For primes $q \notin \pi$, let $\mathscr{X}(q)$ be the empty set. It is not difficult to verify that $\{\mathscr{S}_p\mathscr{X}(p)\}$ is the canonical definition of $\mathscr{X}_\pi$.

We now characterize those saturated formations $\mathscr{F}$ which satisfy condition B.

THEOREM 1. *Let $\mathscr{F}$ be a saturated formation with characteristic π. Let $\Sigma = \pi \cup \{2, 3\}$. Then $\mathscr{F}$ satisfies condition* B *if and only if the following two conditions are satisfied*:

(a) *π is a special set of primes.*

(b) *$\mathscr{F}$ is a subgroup-closed subformation of $\mathscr{X}_\Sigma$ and either $\mathscr{F} \subseteq \mathscr{Y}_\pi$ or $S_3 \in \mathscr{F}$.*

We divide the proof of this theorem into the following two lemmas.

LEMMA 1. *Let $\mathscr{F}$ be a saturated formation satisfying condition* B *and let π be the characteristic of $\mathscr{F}$. Then*:

(a) *π is a special set of primes.*

(b) *Let $\Sigma = \pi \cup \{2, 3\}$. Then $\mathscr{F}$ is a subgroup-closed subformation of $\mathscr{X}_\Sigma$ and either $S_3 \in \mathscr{F}$ or $\mathscr{F} \subseteq \mathscr{Y}_\pi$.*

PROOF. Let $p \in \pi$, p odd. Then $C_p \in \mathscr{F}$, hence Aut $C_p \cong C_{p-1} \in \mathscr{F}$. Since $\mathscr{F}$ is subgroup-closed, it follows that π is a special set of primes.

Let $H \in \mathscr{F}$ and let L/K be a chief factor of H. Let $|L/K| = p^n$, $n \geqq 1$ and $p \in \pi$. Since $\mathscr{F}$ consists of soluble groups only, by Theorem 8.27 of [8], $n \leqq 2$. Thus, since L/K was an arbitrary chief factor of H, we have $r_p(H) \leqq 2$ for all primes p dividing $|H|$. Moreover, by Theorem 2.8.3 of [3], we have $r_p(H) \leqq 1$ if $p \geqq 5$. In fact, $r_p(H) \leqq 1$ if $p \geqq 3$. For assume that $|L/K| = 9$. Then S_4, the symmetric group on four letters, belongs to $\mathscr{F}$. But then, the group G in Example 1 of Kegel [7] also belongs to $\mathscr{F}$ since $G/X' \cong S_4 \times S_4$ and $X' \subseteq \phi(G)$. However, X' is a chief factor of G of order

2^6 and $GL(6, 2)$ being a nonsoluble group does not belong to $\mathscr{F}$. Thus, we have arrived at a contradiction, and so $r_3(H) \leqq 1$.

Next we show that H' is nilpotent. Let p be a prime divisor of $|H|$. If $p \geqq 3$, then $r_p(G) \leqq 1$, hence $H' \subseteq O_{p'p}(H)$. Assume that $p=2$. Then $H' \subseteq O_{2'2}(H)$. For assume that this is not the case. Then some chief factor L/K of H has order 4 and its automizer $H/C_H(L/K) \cong S_3$. But then, by Hilfssatz 7.21 of Huppert [**4**, p. 707], the semidirect product $[L/K]H/C_H(L/K)$ of L/K by $H/C_H(L/K)$ belongs to $\mathscr{F}$, whence $S_4 \in \mathscr{F}$, a contradiction. Thus, $H' \subseteq O_{p'p}(H)$ for each prime divisor p of H. Therefore, $H' \subseteq F(H)$, the Fitting subgroup of H, because of Satz 4.3 of [**4**, p. 278] and Satz 5.4 of [**4**, p. 686]. Hence, H' is nilpotent.

Finally, if $\mathscr{F} \nsubseteq \mathscr{Y}_\pi$, then there exists a group $G \in \mathscr{F}$ such that G has a 2-chief factor of order 4, whence $S_3 \in \mathscr{F}$. Hence the lemma follows.

LEMMA 2. *Let $\mathscr{F}$ be a saturated formation which is subgroup-closed. Let π be the characteristic of $\mathscr{F}$ and let $\Sigma = \pi \cup \{2, 3\}$. Assume that π is a special set of primes and that $\mathscr{F}$ is a subformation of $\mathscr{X}_\Sigma$. If either $\mathscr{F} \subseteq \mathscr{Y}_\pi$ or $S_3 \in \mathscr{F}$, then $\mathscr{F}$ satisfies condition* B.

PROOF. If $\mathscr{F} \subseteq \mathscr{Y}_\pi$, then clearly $\mathscr{F}$ satisfies condition B. Thus assume that $\mathscr{F} \nsubseteq \mathscr{Y}_\pi$ and $S_3 \in \mathscr{F}$. Let $G \in \mathscr{F}$ and let N be a minimal normal subgroup of G. If N is cyclic, then $\mathrm{Aut}(N) \in \mathscr{F}$ since $\pi = \Sigma$ is special. If $|N| = 4$, then $\mathrm{Aut}(N) \cong S_3 \in \mathscr{F}$. Therefore, in either case $\mathrm{Aut}(N) \in \mathscr{F}$, and so $\mathscr{F}$ satisfies condition B.

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