

## EXTENSIONS OF MEASURES AND THE VON NEUMANN SELECTION THEOREM<sup>1</sup>

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**ABSTRACT.** Let  $(X, B_X)$  be a Blackwell space, where  $B_X$  is the  $\sigma$ -algebra of Borel sets. Then if  $\sigma$  is a finite measure defined on a countably generated sub- $\sigma$ -algebra  $B \subset B_X$ ,  $\sigma$  can be extended to a Borel measure  $\tau$ . Equivalently, if  $X$  and  $Y$  are Blackwell and  $f: X \rightarrow Y$  is Borel, and  $\mu$  is a Borel measure carried on  $f(X) \subset Y$ , then there exists a Borel measure  $\tau$  on  $X$  with  $\tau' = \sigma$ , where  $\tau'(E) = \tau(f^{-1}(E))$ . We characterize  $\{\tau | \tau' = \sigma\}$  if  $f$  is semischlicht.

Let  $B_X$  denote the Borel sets of a topological space  $X$ . We consider the following measure extension (or equivalently restriction) problem: given a measure (we will always mean finite measure)  $\sigma$  defined on a  $\sigma$ -algebra  $B \subset B_X$ , can  $\sigma$  be extended to all of  $B_X$ , i.e., does there exist a Borel measure  $\tau$  such that  $\tau(E) = \sigma(E)$  for all  $E \in B$ ? It is well known (see [1, p. 71], for details) that if  $B_1$  and  $B_2$  are  $\sigma$ -algebras, and  $B_2$  is generated by  $B_1$  and finitely many additional sets, then any measure on  $B_1$  can be extended to  $B_2$ . The result is not known for countably generated extensions. We show below (Theorem 5) that if  $X$  is a Blackwell space and  $B$  is a countably generated sub- $\sigma$ -algebra of  $B_X$ , then any measure on  $B$  extends to  $B_X$ .

A Blackwell space is a measure space  $(X, B_X)$ , where  $X$  is an analytic subset of a complete separable metric space (c.s.m.). A subset  $A$  of a c.s.m. is analytic iff  $A$  is the continuous image of a c.s.m. We note that the analytic sets form a proper subset of  $U_X$ , the set of absolutely measurable subsets of  $X$ , where  $E \in U_X$  iff  $E$  is  $\bar{\mu}$ -measurable for all finite Borel measures  $\mu$ , where " $\bar{\mu}$ " denotes the completion of  $\mu$ , i.e., given  $\mu$ , there exist  $E_1, E_2 \in B_X$  such that  $E_1 \subset E \subset E_2$  and  $\mu(E_2 - E_1) = 0$ . A function  $g$  is said to be absolutely measurable if  $g^{-1}(V) \in U_X$  for all open  $V$ . Details may be found in [3], [4], or [5]. We note that if  $X \subset S$ ,  $X$  analytic,  $S$  a c.s.m., then  $B_X = \{E \cap X | E \in B_S\}$ , so elements of  $B_X$  are topologically analytic, and not necessarily Borel in  $S$ .

We begin by considering a special class of sub- $\sigma$ -algebras of  $B_X$ . Let  $f: X \rightarrow Y$  be Borel measurable, and let  $B_f = \{f^{-1}(E) | E \in B_Y\}$ . Given a Borel

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measure  $\tau$  on  $X$ , let  $\tau^f$  be the Borel measure in  $Y$  defined by  $\tau^f(E) = \tau(f^{-1}(E))$ . It follows from a well known isomorphism theorem (see [7, p. 270]) that if  $\tau$  is a positive Borel measure on a c.s.m.  $X$ , and  $\mu$  is a positive Borel measure on a c.s.m.  $Y$  with  $\mu(Y) = \tau(X)$  and  $\tau(\{x\}) = 0$  for all  $x \in X$ , then there exists a Borel function  $f: X \rightarrow Y$  with  $\tau^f = \mu$ . The following shows that a dual of this is equivalent to our extension problem.

**PROPOSITION 1.** *Let  $f: X \rightarrow Y$  be Borel measurable. Then every measure on  $B_f$  is the restriction of a Borel measure iff for every  $\mu$  defined on  $B_Y$  with  $\bar{\mu}(f(X)^c) = 0$  (i.e.,  $\mu$  is carried on  $f(X)$ ), there exists  $\tau$  on  $B_X$  with  $\tau^f = \mu$ .*

**PROOF.** Assume that all Borel  $\mu$  carried on  $f(X) \subset Y$  are of the form  $\mu = \tau^f$ , and let  $\sigma$  be a measure on  $B_f \subset B_X$ . Define  $\mu$  on  $B_Y$  by  $\mu(E) = \sigma(f^{-1}(E))$ ,  $E \subset Y$ . By assumption, there exists a Borel measure  $\tau$  with  $\tau^f = \mu$ . Thus,  $\tau(f^{-1}(E)) = \mu(E) = \sigma(f^{-1}(E))$ , so  $\tau$  extends  $\sigma$ .

Conversely, suppose the extension property holds. Then given a Borel measure  $\mu$  carried on  $f(X) \subset Y$ , define  $\sigma$  on  $B_f$  by  $\sigma(f^{-1}(E)) = \mu(E)$ . (Note  $\sigma$  is well-defined since  $\mu$  is carried on  $f(X)$ .) By assumption,  $\sigma$  is the restriction of some  $\tau$  defined on  $B_X$ , and clearly  $\tau^f = \mu$ .

The proof of our main result relies on the following “selection” theorem of von Neumann.

**THEOREM 2 [6].** *Let  $A$  be an analytic subset of a c.s.m.  $S$ , and let  $F$  be a continuous real-valued function on  $A$ . Then there exists an absolutely measurable  $G: F(A) \rightarrow A$  such that  $F \circ G$  is the identity on  $F(A)$ .*

**COROLLARY 3.** *If  $X$  and  $Y$  are Blackwell and  $F: X \rightarrow Y$  is Borel, there exists an absolutely measurable  $G: Y \rightarrow X$  such that  $F \circ G$  is the identity on  $F(X)$ .*

**PROOF.** Since  $Y$  is Blackwell, we may assume  $Y \subseteq \mathbb{R}$ , the real line. Let  $(F \times I): (X \times Y) \rightarrow (Y \times X)$  be defined by  $(F \times I)((x, y)) = (F(x), y)$ , and let  $P_j$  be projection on the  $j$ th coordinate,  $j = 1, 2$ . Then

$$A = \{(x, F(x)) \mid x \in X\} = (F \times I)^{-1}(\{(y, y)\})$$

is Borel in  $X \times Y$ , and is thus an analytic subset of a c.s.m. Since  $P_2: A \rightarrow Y$  is continuous, there exists an absolutely measurable  $g: P_2(A) \rightarrow A$  with  $P_2 \circ g$  the identity on  $P_2(A) = F(X)$ . Then  $G = P_1 \circ g$  satisfies the theorem.

**REMARK 4.** (i) If  $f: X \rightarrow Y$  is Borel and  $B_Y$  is countably generated, then  $B_f$  is countably generated.

(ii) If  $X$  is Blackwell and  $B \subset B_X$  is countably generated, then there exists a Borel  $f: X \rightarrow \mathbb{R}$  with  $B = B_f$ . (This was pointed out to me by the referee.)

**PROOF.** (i) follows immediately.

Suppose  $\{E_n\}$  generates the  $\sigma$ -algebra  $B$ . Consider  $f(x) = \sum_n 3^{-n} \chi_{E_n(x)}$ . Then since  $B$  and  $B_f$  clearly have the same atoms and  $X$  is Blackwell, it follows that  $B = B_f$  (see [5, p. 38]).

**THEOREM 5.** *Let  $X$  be Blackwell and let  $\sigma$  be a finite measure on a countably generated sub- $\sigma$ -algebra  $B$ . Then  $\sigma$  has an extension  $\tau$  to the full  $\sigma$ -algebra  $B_X$ .*

We first prove the following

**LEMMA.** *If  $f: X \rightarrow Y$  is absolutely measurable and  $S \subset Y$  is absolutely measurable, then  $f^{-1}(S) \subset X$  is absolutely measurable. In particular, a composition of absolutely measurable functions is absolutely measurable.*

**PROOF.** Given any  $\mu$  on  $B_X$ , consider  $\nu = (\bar{\mu})^f$  defined on  $B_Y$ . Since  $S$  is absolutely measurable, there exist  $S_1, S_2 \in B_Y$  with  $S_1 \subset S \subset S_2$ ,  $\nu(S_2 - S_1) = 0$ . Let  $T_i = f^{-1}(S_i)$ . Then  $T_1 \subset f^{-1}(S) \subset T_2$ ,  $T_1, T_2$  are absolutely measurable, and  $\bar{\mu}(T_2 - T_1) = \bar{\mu}(f^{-1}(S_2 - S_1)) = \nu(S_2 - S_1) = 0$ . Thus,  $f^{-1}(S)$  is  $\bar{\mu}$ -measurable.

**PROOF OF THEOREM 5.** Given  $\sigma$  on  $B$ , we have  $B = B_f$  where  $f: X \rightarrow \mathbf{R}$  is Borel. Then there exists an absolutely measurable  $g: f(X) = Y \rightarrow X$  with  $f \circ g = i_Y$ . Define  $\tau$  on  $B_X$  by  $\tau(S) = \bar{\sigma}((g \circ f)^{-1}(S))$ . (This makes sense, since  $g \circ f$  is absolutely measurable.) Clearly, if  $S \in B_f$ , there exists  $T \subset Y$  such that  $S = f^{-1}(T)$  and  $T \in B_Y$ .

Then

$$\begin{aligned} \tau(S) &= \bar{\sigma}((g \circ f)^{-1}(S)) = \bar{\sigma}(f^{-1}(g^{-1}(S))) \\ &= \bar{\sigma}(f^{-1}((f \circ g)^{-1}(T))) = \bar{\sigma}(f^{-1}(T)) \\ &= \bar{\sigma}(S) = \sigma(S). \end{aligned}$$

Thus,  $\tau$  extends  $\sigma$  to  $B_X$ .

**COROLLARY 6.** *Let  $f: X \rightarrow Y$  be Borel, where  $X$  and  $Y$  are Blackwell, and let  $\mu$  be a Borel measure carried on  $f(X) \subset Y$ . Then there exists  $\tau$  on  $B_X$  with  $\tau^f = \mu$ .*

The proof of the corollary is explicit in the proof of the theorem. We note that the  $\tau$  obtained is of the form  $\tau(E) = \mu(g^{-1}(E)) = \mu(f(E \cap S))$ , where  $S = g(Y)$ .  $S$  thus consists of one point "selected" from each pre-image set  $f^{-1}(\{y\})$ . Hence,  $\tau$  is carried on a 1-1 set, i.e., a section, of  $f$ .

If  $f$  is 1-1, then  $\tau$  is clearly unique. Suppose that  $f$  is semischlicht, i.e., that  $f^{-1}(\{y\})$  is a countable set for all  $y \in Y$ . Then there is a countable collection of disjoint Borel sets  $\{S_n\}$  such that  $\bigcup S_n = X$ ,  $f_n = f|_{S_n}$  is 1-1, and  $f(S_n) = f(X - \bigcup_{j < n} S_j)$ . (See [2, p. 335].) Given a Borel measure  $\sigma$  on  $Y$ , let  $\tau_n$  be defined on  $B_X$  by  $\tau_n(E) = \sigma(f(E \cap S_n))$ . Then  $\tau_n^f = \sigma|_{f(S_n)}$ .

Choose  $a_n \in L^1(\tau_n)$  such that  $\sum_n a_n(f_n^{-1}(y))=1$  a.e.  $[\sigma]$ , where

$$a_n(f_n^{-1}(y)) = 0 \quad \text{if } y \notin f_n(S_n),$$

and the series is absolutely convergent. Let  $\tau$  be defined on  $B_X$  by  $\tau(E)=\sum_n \int a_n(x) d\tau_n(x)$ , i.e.,  $d\tau(x)=\sum_n a_n(x) d\tau_n(x)$  so  $\tau$  is a "convex combination" of the measures  $\tau_n$ . Then it is easy to see that  $\tau^f=\sigma$ , and we see below that this characterizes  $\{\tau|\tau^f=\sigma\}$ .

**PROPOSITION 7.** *Let  $X$  and  $Y$  be Blackwell, and let  $f: X \rightarrow Y$  be a semischlicht Borel map. Let  $\sigma$  be a measure defined on  $B_Y$ , and suppose that  $\{S_n\}$ ,  $\{f_n\}$  and  $\{\tau_n\}$  are as above. Then there exists  $\{a_n(x)\}$  such that  $a_n \in L^1(\tau_n)$ ,  $\sum_n a_n(f_n^{-1}(y))=1$  a.e.  $[\sigma]$ , and  $d\tau(x)=\sum_n a_n(x) d\tau_n(x)$ .*

**PROOF.** Using the Hahn decomposition, we may assume that  $\sigma$  and  $\tau$  are positive. Then for  $E \subset S_n \subset X$ ,

$$\tau(E) = \tau(f_n^{-1}(f_n(E))) \leq \tau(f^{-1}(f_n(E))) = \sigma(f_n(E)) = \tau_n(E),$$

so by the Radon-Nikodym theorem, there exists  $a_n \in L^1(\tau_n)$  with

$$\tau(E) = \int_E a_n(x) d\tau_n(x) = \int_{f_n(E)} a_n(f_n^{-1}(y)) d\sigma(y) \quad \text{if } E \subset S_n.$$

Thus, for  $E \subset X$ ,

$$\tau(E) = \sum_n \tau(E \cap S_n) = \int_E \sum_n a_n(x) d\tau_n(x),$$

so

$$d\tau(x) = \sum_n a_n(x) d\tau_n(x).$$

For  $E \subset Y$ ,

$$\begin{aligned} \sigma(E) &= \tau(f^{-1}(E)) = \sum_n \int_{f_n(f^{-1}(E) \cap S_n)} a_n(f_n^{-1}(y)) d\sigma(y) \\ &= \int_E \left( \sum_n a_n(f_n^{-1}(y)) \right) d\sigma(y). \end{aligned}$$

Hence,

$$\sum_n a_n(f_n^{-1}(y)) = 1 \text{ a.e. } [\sigma].$$

We point out that if, in fact,  $\tau(E)=\sigma(f(E \cap S))$  for  $E \subset X$ , where  $f|_S$  is 1-1, then for each  $n$ , there exists  $D_n \subset S_n$  such that  $a_n = \chi_{D_n}$  and  $S = \bigcup_n D_n$ . Further, the following example shows that the assumption  $f$  semischlicht cannot be dropped.

For  $a \in [0, 1]$  an irrational number, consider the unique decimal representation  $a=0.a_1a_2 \dots$ , and let  $f: [0, 1] \rightarrow [0, 1]$  be defined by  $f(0.a_1a_2a_3 \dots) = 0.a_1a_3a_5 \dots$ . Then, after defining  $f$  in an appropriate manner on the rational points, which we can ignore since we will be using

nonatomic measures,  $f$  maps  $[0, 1]$  onto  $[0, 1]$ , and

$$f^{-1}(0.a_1a_2 \cdots) = \{0.a_1x_1a_2x_2 \cdots \mid x_i \text{ arbitrary}\}.$$

Let  $\Phi: [0, 1] \rightarrow ([0, 1] \times [0, 1])$  be defined by

$$\Phi(0.a_1a_2a_3 \cdots) = (0.a_1a_3a_5 \cdots, 0.a_2a_4a_6 \cdots).$$

Let  $m$  be Lebesgue measure in  $[0, 1]$ . Let  $\sigma$  be a singular nonatomic measure on  $[0, 1]$ . We now define  $\tau$  on  $[0, 1]$  by  $\tau(E) = (\sigma \times m)(\Phi(E))$ . We note that  $\Phi(E)$  is measurable since  $\Phi$  restricted to the irrationals is a homeomorphism.

Suppose  $E = f^{-1}(B)$  for some Borel set  $B \subset [0, 1]$ . Then

$$E = \{0.a_1x_1a_2x_2 \cdots \mid 0.a_1a_2 \cdots \in B\} \quad \text{and} \quad \Phi(E) = B \times [0, 1].$$

Thus,

$$\begin{aligned} \tau(E) &= \tau(f^{-1}(B)) = (\sigma \times m)(\Phi(E)) \\ &= (\sigma \times m)(B \times [0, 1]) = \sigma(B). \end{aligned}$$

Hence,  $\tau^f = \sigma$ , and it is easy to see that if  $f|_S$  is 1-1, then  $\tau(S) = 0$ , so  $\tau$  cannot be written as a "combination" of measures supported on 1-1 sets of  $f$ .

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