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# HARMONIC NULL SETS AND THE PAINLEVÉ THEOREM

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ABSTRACT. A less restrictive condition on an open Riemann surface than has been formerly known for a subset of the ideal boundary of a resolutive compactification to have harmonic measure zero is demonstrated. Then a generalized version of a classical theorem of Painlevé is established in this framework.

Recently, Arsove and Leutwiler [**2**] proved a generalization of the classical theorem of Painlevé which states that an analytic function on a Jordan region which tends to zero at each point of a nondegenerate boundary arc vanishes identically. Their result stemmed from an important new characterization of harmonic null sets. It is well known [**1**] that the existence of a *positive* harmonic function on a bounded region which tends to $\infty$ on a boundary set $E$ is necessary and sufficient for $E$ to have harmonic measure zero. In [**2**], the requirement of positiveness is disposed of.

A similar situation has existed in the field of potential theory on Riemann surfaces. The existence of a *positive* superharmonic function on $R$ whose lim inf tends to $\infty$ at each point of a subset of a suitable ideal boundary, has long been accepted as a sufficient condition for the subset to have harmonic measure zero. However, it can be demonstrated that here also *positiveness is not required*, and Painlevé's theorem can be extended to Riemann surfaces as well.

THEOREM. *Let $R^*$ be a resolutive compactification of an open Riemann surface $R$, and $\Delta=R^*-R$. If a superharmonic function $s$ on $R$ tends to $\infty$ at all points of $E\subset\Delta$, then the harmonic measure $\omega$ of $E$ is zero.*

PROOF. Let $G=\{z\in R|s(z)>0\}$. Then $G$ is open in $R$, and set

$$A=\left\{\zeta\in\Delta-\mathrm{cl}(R-G)\,\middle|\,\liminf_{z\to\zeta} s(z)=\infty\right\}.$$

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By Hilfssatz 8.8 of [3], $\omega(A)=0$. For any $\zeta\in E$, if $\zeta\notin A$, then we must have $\zeta\in \mathrm{cl}(R-G)$ since $\lim_{z\to\zeta} s(z)=\infty$. Hence there exists a net $\{z_\alpha\}\subset R-G$ such that $z_\alpha\to\zeta$. But $\zeta\in E$ implies that $s(z_\alpha)\to\infty$, which contradicts the fact that $s(z_\alpha)\leqq 0$. It follows that $E\subset A$ and $\omega(E)=0$ as desired.

The theorem is also valid in the theory of harmonic spaces.

Painlevé's classical result can now be generalized to the following:

COROLLARY. *Let $f$ be analytic on $R$, $f\to 0$ at all points of $E\subset\Delta$, where $\omega(E)>0$. Then $f\equiv 0$.*

PROOF. If $f\not\equiv 0$, let $u=-\log|f|$. Then $u$ is superharmonic on $R$, and $u\to\infty$ at all points of $E$. Thus $\omega(E)=0$, a contradiction.

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