ON THE RANGE OF A HOMOMORPHISM OF A GROUP ALGEBRA INTO A MEASURE ALGEBRA

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Abstract. It is shown, that if $G$ is a LCA group and if $H$ is a nondiscrete LCA group then there exists a proper closed subalgebra of the measure algebra of $H$ (independent of the choice of $G$) in which the range of every homomorphism of the group algebra of $G$ into the measure algebra of $H$ is contained.

Throughout this paper, $G$ and $H$ denote LCA groups and $\hat{G}$ and $\hat{H}$ denote their dual groups, respectively. $\mathcal{I}(H)$ is the set of all the locally compact group topologies of $H$ which are at least as strong as the original one of $H$. For each $\tau \in \mathcal{I}(H)$, if we denote by $H^{\tau}$ a LCA group with underlying group $H$ and topology $\tau$, the natural continuous isomorphism of $H^{\tau}$ onto $H$, $x \in H^{\tau} \mapsto x \in H$, induces a natural norm-preserving imbedding of $L^1(H^{\tau})$ into $M(H)$, which we also denote by $L^1(H^{\tau})$. For the other notations and terminologies which we need in this paper, we follow [6].

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Theorem. If $h$ is a homomorphism of $L^1(G)$ into $M(H)$, then there exist finitely many elements $\tau_1, \tau_2, \cdots, \tau_n \in \mathcal{I}(H)$ such that the range of $h$ is contained in $\sum_{i=1}^{n} L^1(H^{\tau_i})$.

For the proof of the theorem, we essentially use Cohen's results, which determine all the homomorphisms of $L^1(G)$ into $M(H)$ by the notion of the coset ring and piecewise affine maps (cf. [1], [2], [3] and [6, Chapters 3 and 4]).

If $h$ is a homomorphism of $L^1(G)$ into $M(H)$, Cohen's theorem asserts that there exist $Y$, an element of the coset ring of $\hat{H}$, and a piecewise affine map $\alpha$ from $Y$ into $\hat{G}$ such that

$$h(f)^\wedge(r) = f(\alpha(r)), \quad r \in Y, \quad f \in L^1(G), \quad r \notin Y$$

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and conversely, if \( Y \) is an element of the coset ring of \( \hat{H} \) and if \( \alpha \) is a piecewise affine map from \( Y \) into \( \hat{G} \), the pair \((Y, \alpha)\) induces a unique homomorphism \( h \) of \( L^1(G) \) into \( M(H) \) which satisfies (1). We call the pair \((Y, \alpha)\) the dual map of \( h \) after P. Eymard [3] (though slightly different from his definition).

For the rest of this paper, \( h \) denotes a homomorphism of \( L^1(G) \) into \( M(H) \) and \((Y, \alpha)\) denotes the dual map of \( h \).

**Lemma 1.** If \( Y \) is an open subgroup of \( \hat{H} \) and \( \alpha(Y) \) is a continuous homomorphism from \( Y \) into \( \hat{G} \), then the range of \( h \) is contained in \( L^1(H^\tau) \) for some \( \tau \in \mathcal{I}(H) \).

**Proof.** We suppose first that \( Y=\hat{H} \) and \( \alpha(Y) \) is dense in \( \hat{G} \), and then there exists a natural continuous isomorphism \( \hat{\alpha} \) of \( G \) into \( H \) such that

\[
(\hat{\alpha}(x), r) = (x, \alpha(r)) \quad (x \in G, r \in \hat{H}).
\]

We can introduce in \( H \) a locally compact group topology \( \tau \) such that \( \hat{\alpha} \) becomes an open continuous map of \( G \) into \( H^\tau \), and then \( \hat{\alpha} \) induces the natural isomorphism of \( L^1(G) \) into \( L^1(H^\tau) \), which just coincides with \( h \).

Next we suppose only that \( \alpha(Y) \) is dense in \( \hat{G} \). By the above considerations, we have an element \( \bar{\tau} \in \mathcal{I}(H/L) \) and a continuous isomorphism \( \hat{h} \) of \( L^1(G) \) into \( L^1((H/L)^\bar{\tau}) \) such that the dual map of \( \hat{h} \) is \((Y, \alpha)\), where \( L \) denotes the annihilator of \( Y \) in \( H \).

Let \( \pi \) be the natural map of \( H \) onto \( H/L \). If we introduce in \( H \) a topology \( \tau \) with a basis \( \{\pi^{-1}(V) \cap W : V \) is open in \((H/L)^{\bar{\tau}} \) and \( W \) is open in \( H \} \), then \( \tau \) is a locally compact group topology of \( H \), and the map \( \pi \) induces an open continuous homomorphism of \( H^\tau \) onto \((H/L)^\bar{\tau}\). For each \( f \in L^1((H/L)^\bar{\tau}) \), put \( h'(f) = f \circ \pi \), then \( h'(f) \) belongs to \( L^1(H^\tau) \) and \( h'h = h \). Thus we have \( h(L^1(G)) = h'h(L^1(G)) \subset L^1(H^\tau) \).

Finally we prove the general case. Let \( \Lambda \) be the closure of \( \alpha(Y) \) in \( \hat{G} \). Then there exists a homomorphism \( h'' \) of \( L^1(G/K) \) into \( M(H) \) with the dual map \((Y, \alpha)\), where \( K \) is the annihilator of \( \Lambda \) in \( G \). Since \( A(\Lambda) \) coincides with the set \( \{f|_{\Lambda} : f \in A(\hat{G})\} \), we can reduce the problem to the preceding case; thus we have \( h(L^1(G)) = h''(L^1(G/K)) \subset L^1(H^\tau) \) for some \( \tau \in \mathcal{I}(H) \). This completes the proof.

**Lemma 2.** If \( Y \) is an open coset and \( \alpha \) is an affine map, then we get the same conclusion as Lemma 1.

**Proof.** Let \( r_2 \) be an element of \( H \) such that \( Y-r_2 \) is an open subgroup of \( \hat{H} \). There exist a continuous homomorphism \( \beta \) of \( Y-r_2 \) into \( \hat{G} \) and \( r_1 \in \hat{G} \) such that

\[
\alpha(r) = \beta(r - r_2) - r_1 \quad (r \in Y).
\]
By Lemma 1, there exist an element \( \tau \in \mathfrak{I}(H) \) and a continuous homomorphism \( h' \) of \( L^1(G) \) into \( L^1(H') \) with the dual map \( Y - \tau_2, \beta \). If we define \( h_1 \) and \( h_2 \) by

\[
    h_1(f) = r_1 f \quad (f \in L^1(G)); \quad h_2(g) = r_2 g \quad (g \in L^1(H')),
\]
then \( h_1 \) and \( h_2 \) are homomorphisms of \( L^1(G) \) into \( L^1(G) \) and \( L^1(H') \) into \( L^1(H') \), respectively. Since \( h = h_2 h' h_1 \), the range of \( h \) is contained in \( L^1(H') \) and Lemma 2 is proved.

Let \( J(H) \) be the set of all the idempotent measures in \( M(H) \), and for each \( \mu \in J(H) \) we put \( S(\mu) = [r \in \hat{H} : \mu(r) = 1] \).

**Lemma 3.** If \( \mu \) is an element of \( J(H) \), then there exist finitely many compact subgroups \( K_1, K_2, \ldots, K_n \) of \( H \) such that

(i) \( m_{K_i} \) and \( m_{K_j} \) are mutually singular for \( i \neq j \),
(ii) for \( i \) and \( j \), we have \( m_{K_i} \ast m_{K_j} = m_{K_i + K_j} \ll m_{K_1} \) (absolutely continuous with respect to \( m_{K_1} \)) for some \( l \),
(iii) \( \mu \ll \sum_{i=1}^n m_{K_i} \),

where \( m_K \) denotes the Haar measure of a compact group \( K \).

**Proof.** There exists a set \([K_1, K_2, \ldots, K_m]\) of finitely many compact subgroups of \( H \) which satisfies the conditions (i) and (ii) (cf. [5]). We can choose finitely many compact subgroups \( K_{m+1}, \ldots, K_n \) of \( H \) (if necessary) so that \([K_1, K_2, \ldots, K_n]\) satisfies the conditions (i), (ii) and (iii), and this completes the proof.

**Lemma 4.** If there exist an open coset \( \Lambda \) and an affine map \( \tilde{\alpha} \) of \( \Lambda \) into \( \hat{G} \) such that \( Y = \Lambda, \tilde{\alpha}|_Y = \alpha \), then we get the conclusion of the theorem.

**Proof.** Since \( Y \) is an element of the coset ring of \( H \), there exists \( \mu \in J(H) \) such that \( S(\mu) = Y \). Since \( \mu \) is determined by \( h \), we express \( \mu \) by \( j(h) \). Let \([K_1, K_2, \ldots, K_n]\) be a set of finitely many compact subgroups of \( H \) which satisfies (i), (ii) and (iii) of Lemma 3. We decompose \( \mu = \lambda_1 + \lambda_2 + \cdots + \lambda_n \), \( \lambda_i \ll m_{K_i} \) \( (i = 1, 2, \ldots, n) \), and we proceed by induction on the number \( n \) of \([K_1, K_2, \ldots, K_n]\). Thus we suppose that Lemma 4 is true if \( n \leq k \), and prove that Lemma 4 is also true for \( n = k + 1 \).

We can suppose without loss of generality that \( K_n \) is minimal in the sense that \( K_i / K_n \cap K_i \) is infinite for \( i \neq n \). Then since \( \mu = \mu \ast \mu = \lambda_n \ast \lambda_n + \sum_{i \neq n \text{ or } j \neq n} \lambda_i \ast \lambda_j \), we get \( \sum_{i \neq n \text{ or } j \neq n} \lambda_i \ast \lambda_j \ll \sum_{i=1}^{n-1} m_{K_i} \) and \( \lambda_n \in J(H) \).

If we put

\[
    h_1 : f \in L^1(G) \mapsto h(f) \ast \lambda_n \ast \mu \in M(H),
    h_2 : f \in L^1(G) \mapsto h(f) \ast (\mu - \mu \ast \lambda_n) \in M(H),
\]
then $h_1$ and $h_2$ are homomorphisms which satisfy $h_1(f) + h_2(f) = h(f)$ ($f \in L^1(G)$). Since $[K_1, K_2, \cdots, K_{n-1}]$ satisfies the conditions (i), (ii) and (iii) of Lemma 3 for $\mu = j(h_2)$, we have by the assumption of the induction that $h_2(L^1(G)) \subseteq \bigcup_{i \in I} L^1(H')$ for some finite subset $A \subseteq \mathcal{X}(H)$, and we have only to prove the lemma for $h = h_1$. Therefore we can assume here without loss of generality that $\lambda_n \ast \mu = \mu$, that is $S(\lambda_n) \subseteq S(\mu)$. Obviously, $\lambda_n$ is an irreducible idempotent, and hence there exist $r_1, r_2, \cdots, r_m \in \hat{H}$ such that $d\lambda_n = [(x, r_1) + \cdots + (x, r_m)] dm_{K_n}$, where $r_i - r_j$ ($i \neq j$) does not belong to the annihilator of $K_n$.

For each $i$, let $\sigma_i$ be an element of $J(H)$ such that $d\sigma_i = (x, r_i) dm_{K_n}$ and let $h_i$ be a homomorphism of $L^1(G)$ into $M(H)$ with the dual map $(S(\sigma_i) \cap \Lambda, \bar{x}|_{S(\sigma_i) \cap \Lambda})$. Let $h'_i$ and $h''_i$ be homomorphisms of $L^1(G)$ into $M(H)$ such that $h'_i(f) = h(f) \ast \sigma_i$ and $h''_i(f) = h_i(f) \ast (\sigma_i - \sigma_i \ast \mu)$, and then we have $h''_i(f) = h'_i(f) - h''_i(f)$ ($f \in L^1(G)$). By Lemma 2, $h_i$ maps $L^1(G)$ into $L^1(H')$ for some $\tau_i \in \mathcal{X}(H)$, and since $j(h''_i)$ is absolutely continuous with respect to $\sum_{i=1}^m m_{K_i}$, we have again by the assumption of the induction that $h''_i$ maps $L^1(G)$ into $\bigcap_{i \in I} L^1(H')$ for some finite subset $B_i \subseteq \mathcal{X}(H)$, and consequently we get

$$h(L^1(G)) \subseteq \bigcup_{i=1}^m h_i(L^1(G)) - \bigcap_{i=1}^m h''_i(L^1(G)) \subseteq \bigcap_{i \in I} L^1(H'),$$

and this completes the proof.

**The proof of the theorem.** Let $(Y, \alpha)$ be the dual map of $h$. There exist a set of pairwise disjoint elements $\{Y_i\}_{i=1}^n$ of the coset ring of $\hat{H}$, a set of open cosets $\{K_i\}_{i=1}^n$ of $\hat{H}$ and a set of affine maps $\{a_i : K_i \to G\}_{i=1}^n$ such that

$$Y = Y_1 \cup \cdots \cup Y_n, \quad K_i \supseteq Y_i, \quad a_i|_{Y_i} = a_i|_{Y_i} \quad (i = 1, 2, \cdots, n).$$

If we denote by $h_i$ a homomorphism of $L^1(G)$ into $M(H)$ with the dual map $(Y_i, a_i|_{Y_i})$ ($i = 1, 2, \cdots, n$), then we have $h(f) = h_1(f) + \cdots + h_n(f)$ ($f \in L^1(G)$). By Lemma 4 we have $h_i(f) \in \bigcup_{i \in I} L^1(H')$ for some finite subset $A_i \subseteq \mathcal{X}(H)$, and hence $h(f)$ belongs to $\bigcup_{i \in I} \bigcap_{i \in I} L^1(H')$ for each $f \in L^1(G)$ and $i$, and thus the theorem is proved.

**Remark.** If we refer to [4], we can see that $\sum_{\tau \in \mathcal{X}(H)} L^1(H')$ is a subalgebra of $M(H)$ and that the norm closure of $\sum_{\tau \in \mathcal{X}(H)} L^1(H')$ in $M(H)$ is a proper closed subalgebra of $M(H)$ if $H$ is not discrete. This means that the set of the elements of the form $h(x)$ ($x \in G$), where a LCA group $G$ and a homomorphism $h$ of $L^1(G)$ into $M(H)$ vary arbitrarily, constitutes the subalgebra $\sum_{\tau \in \mathcal{X}(H)} L^1(H')$ contained (if $H$ is not discrete) in a proper closed subalgebra of $M(H)$. 

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