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ON THE RANGE OF A HOMOMORPHISM OF A GROUP ALGEBRA INTO A MEASURE ALGEBRA

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ABSTRACT. It is shown, that if G is a LCA group and if H is a nondiscrete LCA group then there exists a proper closed subalgebra of the measure algebra of H (independent of the choice of G) in which the range of every homomorphism of the group algebra of G into the measure algebra of H is contained.

Throughout this paper, G and H denote LCA groups and $\hat{G}$ and $\hat{H}$ denote their dual groups, respectively. $\mathfrak{T}(H)$ is the set of all the locally compact group topologies of H which are at least as strong as the original one of H. For each $\tau \in \mathfrak{T}(H)$, if we denote by H^τ a LCA group with underlying group H and topology τ, the natural continuous isomorphism of H^τ onto H, $x \in H^\tau \mapsto x \in H$, induces a natural norm-preserving imbedding of $L^1(H^\tau)$ into $M(H)$, which we also denote by $L^1(H^\tau)$. For the other notations and terminologies which we need in this paper, we follow [**6**].

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THEOREM. *If h is a homomorphism of $L^1(G)$ into $M(H)$, then there exist finitely many elements $\tau_1, \tau_2, \cdots, \tau_n \in \mathfrak{T}(H)$ such that the range of h is contained in $\sum_{i=1}^n L^1(H^{\tau_i})$.*

For the proof of the theorem, we essentially use Cohen's results, which determine all the homomorphisms of $L^1(G)$ into $M(H)$ by the notion of the coset ring and piecewise affine maps (cf. [**1**], [**2**], [**3**] and [**6**, Chapters 3 and 4]).

If h is a homomorphism of $L^1(G)$ into $M(H)$, Cohen's theorem asserts that there exist Y, an element of the coset ring of $\hat{H}$, and a piecewise affine map α from Y into $\hat{G}$ such that

$$(1) \qquad \begin{aligned} h(f)^\wedge(r) &= \hat{f}(\alpha(r)), & r \in Y \\ &= 0, & r \notin Y \end{aligned} \quad (f \in L^1(G)),$$

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and conversely, if Y is an element of the coset ring of $\hat{H}$ and if α is a piecewise affine map from Y into $\hat{G}$, the pair (Y, α) induces a unique homomorphism h of $L^1(G)$ into $M(H)$ which satisfies (1). We call the pair (Y, α) the dual map of h after P. Eymard [3] (though slightly different from his definition).

For the rest of this paper, h denotes a homomorphism of $L^1(G)$ into $M(H)$ and (Y, α) denotes the dual map of h.

LEMMA 1. *If Y is an open subgroup of $\hat{H}$ and α is a continuous homomorphism from Y into $\hat{G}$, then the range of h is contained in $L^1(H^\tau)$ for some $\tau \in \mathfrak{T}(H)$.*

PROOF. We suppose first that $Y=\hat{H}$ and $\alpha(Y)$ is dense in $\hat{G}$, and then there exists a natural continuous isomorphism $\hat{\alpha}$ of G into H such that

$$(\hat{\alpha}(x), r) = (x, \alpha(r)) \qquad (x \in G, r \in \hat{H}).$$

We can introduce in H a locally compact group topology τ such that $\hat{\alpha}$ becomes an open continuous map of G into H^τ, and then $\hat{\alpha}$ induces the natural isomorphism of $L^1(G)$ into $L^1(H^\tau)$, which just coincides with h.

Next we suppose only that $\alpha(Y)$ is dense in $\hat{G}$. By the above considerations, we have an element $\bar{\tau} \in \mathfrak{T}(H/L)$ and a continuous isomorphism $\bar{h}$ of $L^1(G)$ into $L^1((H/L)^{\bar{\tau}})$ such that the dual map of $\bar{h}$ is (Y, α), where L denotes the annihilator of Y in H.

Let π be the natural map of H onto H/L. If we introduce in H a topology τ with a basis $[\pi^{-1}(V) \cap W : V$ is open in $(H/L)^{\bar{\tau}}$ and W is open in $H]$, then τ is a locally compact group topology of H, and the map π induces an open continuous homomorphism of H^τ onto $(H/L)^{\bar{\tau}}$. For each $f \in L^1((H/L)^{\bar{\tau}})$, put $h'(f)=f \circ \pi$, then $h'(f)$ belongs to $L^1(H^\tau)$ and $h'\bar{h}=h$. Thus we have $h(L^1(G))=h'\bar{h}(L^1(G)) \subset L^1(H^\tau)$.

Finally we prove the general case. Let Λ be the closure of $\alpha(Y)$ in $\hat{G}$. Then there exists a homomorphism h'' of $L^1(G/K)$ into $M(H)$ with the dual map (Y, α), where K is the annihilator of Λ in G. Since $A(\Lambda)$ coincides with the set $[\hat{f}|_\Lambda : f \in A(\hat{G})]$, we can reduce the problem to the preceding case; thus we have $h(L^1(G))=h''(L^1(G/K)) \subset L^1(H^\tau)$ for some $\tau \in \mathfrak{T}(H)$. This completes the proof.

LEMMA 2. *If Y is an open coset and α is an affine map, then we get the same conclusion as Lemma 1.*

PROOF. Let r_2 be an element of H such that $Y - r_2$ is an open subgroup of $\hat{H}$. There exist a continuous homomorphism β of $Y - r_2$ into $\hat{G}$ and $r_1 \in \hat{G}$ such that

$$\alpha(r) = \beta(r - r_2) - r_1 \qquad (r \in Y).$$

By Lemma 1, there exist an element $\tau \in \mathfrak{X}(H)$ and a continuous homomorphism h' of $L^1(G)$ into $L^1(H^\tau)$ with the dual map $(Y - r_2, \beta)$. If we define h_1 and h_2 by

$$h_1(f) = r_1 f \quad (f \in L^1(G)); \qquad h_2(g) = r_2 g \quad (g \in L^1(H^\tau)),$$

then h_1 and h_2 are homomorphisms of $L^1(G)$ into $L^1(G)$ and $L^1(H^\tau)$ into $L^1(H^\tau)$, respectively. Since $h = h_2 h' h_1$, the range of h is contained in $L^1(H^\tau)$ and Lemma 2 is proved.

Let $J(H)$ be the set of all the idempotent measures in $M(H)$, and for each $\mu \in J(H)$ we put $S(\mu) = [r \in \hat{H} : \hat{\mu}(r) = 1]$.

LEMMA 3. *If μ is an element of $J(H)$, then there exist finitely many compact subgroups $K_1, K_2, \cdots, K_n$ of H such that*

(i) *m_{K_i} and m_{K_j} are mutually singular for $i \neq j$,*

(ii) *for i and j, we have $m_{K_i} * m_{K_j} = m_{K_i + K_j} \ll m_{K_l}$ (absolutely continuous with respect to m_{K_l}) for some l,*

(iii) *$\mu \ll \sum_{i=1}^n m_{K_i}$,*

where m_K denotes the Haar measure of a compact group K.

PROOF. There exists a set $[K_1, K_2, \cdots, K_m]$ of finitely many compact subgroups of H which satisfies the conditions (i) and (ii) (cf. [5]). We can choose finitely many compact subgroups $K_{m+1}, \cdots, K_n$ of H (if necessary) so that $[K_1, K_2, \cdots, K_n]$ satisfies the conditions (i), (ii) and (iii), and this completes the proof.

LEMMA 4. *If there exist an open coset Λ and an affine map $\bar{\alpha}$ of Λ into $\hat{G}$ such that $Y \subset \Lambda$, $\bar{\alpha}|_Y = \alpha$, then we get the conclusion of the theorem.*

PROOF. Since Y is an element of the coset ring of $\hat{H}$, there exists $\mu \in J(H)$ such that $S(\mu) = Y$. Since μ is determined by h, we express μ by $j(h)$. Let $[K_1, K_2, \cdots, K_n]$ be a set of finitely many compact subgroups of H which satisfies (i), (ii) and (iii) of Lemma 3. We decompose μ as $\mu = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, $\lambda_i \ll m_{K_i}$ $(i = 1, 2, \cdots, n)$, and we proceed by induction on the number n of $[K_1, K_2, \cdots, K_n]$. Thus we suppose that Lemma 4 is true if $n \leqq k$, and prove that Lemma 4 is also true for $n = k + 1$.

We can suppose without loss of generality that K_n is minimal in the sense that $K_i / K_n \cap K_i$ is infinite for $i \neq n$. Then since $\mu = \mu * \mu = \lambda_n * \lambda_n + \sum_{i \neq n \text{ or } j \neq n} \lambda_i * \lambda_j$, we get $\sum_{i \neq n \text{ or } j \neq n} \lambda_i * \lambda_j \ll \sum_{i=1}^{n-1} m_{K_i}$ and $\lambda_n \in J(H)$. If we put

$$h_1 : f \in L^1(G) \mapsto h(f) * \lambda_n * \mu \in M(H),$$

$$h_2 : f \in L^1(G) \mapsto h(f) * (\mu - \mu * \lambda_n) \in M(H),$$

then h_1 and h_2 are homomorphisms which satisfy $h_1(f)+h_2(f)=h(f)$ $(f\in L^1(G))$. Since $[K_1, K_2, \cdots, K_{n-1}]$ satisfies the conditions (i), (ii) and (iii) of Lemma 3 for $\mu=j(h_2)$, we have by the assumption of the induction that $h_2(L^1(G))\subset\sum_{\tau\in A} L^1(H^\tau)$ for some finite subset $A\subset\mathfrak{T}(H)$, and we have only to prove the lemma for $h=h_1$. Therefore we can assume here without loss of generality that $\lambda_n * \mu=\mu$, that is $S(\lambda_n)\supset S(\mu)$. Obviously, λ_n is an irreducible idempotent, and hence there exist $r_1, r_2, \cdots, r_m\in\hat{H}$ such that $d\lambda_n=[(x, r_1)+\cdots+(x, r_m)]\, dm_{K_n}$, where r_i-r_j $(i\neq j)$ does not belong to the annihilator of K_n.

For each i, let σ_i be an element of $J(H)$ such that $d\sigma_i=(x, r_i)\, dm_{K_n}$ and let h_i be a homomorphism of $L^1(G)$ into $M(H)$ with the dual map $(S(\sigma_i)\cap\Lambda, \bar{\alpha}|_{S(\sigma_i)\cap\Lambda})$. Let h_i' and h_i'' be homomorphisms of $L^1(G)$ into $M(H)$ such that $h_i'(f)=h(f)*\sigma_i$ and $h_i''(f)=h_i(f)*(\sigma_i-\sigma_i*\mu)$, and then we have $h_i'(f)=h_i(f)-h_i''(f)$ $(f\in L^1(G))$. By Lemma 2, h_i maps $L^1(G)$ into $L^1(H^{\tau_i})$ for some $\tau_i\in\mathfrak{T}(H)$, and since $j(h_i'')$ is absolutely continuous with respect to $\sum_{j=1}^{n-1} m_{K_j}$ we have again by the assumption of the induction that h_i'' maps $L^1(G)$ into $\sum_{\tau\in B_i} L^1(H^\tau)$ for some finite subset $B_i\subset\mathfrak{T}(H)$, and consequently we get

$$h(L^1(G))\subset\sum_{i=1}^m h_i(L^1(G))-\sum_{i=1}^m h_i''(L^1(G))\subset\sum_{\tau\in(\bigcup_{i=1}^m B_i)\cup\{\tau_1,\cdots,\tau_m\}} L^1(H^\tau),$$

and this completes the proof.

THE PROOF OF THE THEOREM. Let (Y, α) be the dual map of h. There exist a set of pairwise disjoint elements $\{Y_i\}_{i=1}^n$ of the coset ring of $\hat{H}$, a set of open cosets $\{K_i\}_{i=1}^n$ of $\hat{H}$ and a set of affine maps $\{\alpha_i: K_i\to\hat{G}\}_{i=1}^n$ such that

$$Y=Y_1\cup\cdots\cup Y_n,\quad K_i\supset Y_i,\quad \alpha|_{Y_i}=\alpha_i|_{Y_i}\qquad (i=1,2,\cdots,n).$$

If we denote by h_i a homomorphism of $L^1(G)$ into $M(H)$ with the dual map $(Y_i, \alpha|_{Y_i})$ $(i=1,2,\cdots,n)$, then we have $h(f)=h_1(f)+\cdots+h_n(f)$ $(f\in L^1(G))$. By Lemma 4 we have $h_i(f)\in\sum_{\tau\in A_i} L^1(H^\tau)$ for some finite subset $A_i\subset\mathfrak{T}(H)$, and hence $h(f)$ belongs to $\sum_{\tau\in\bigcup_{i=1}^n A_i} L^1(H^\tau)$ for each $f\in L^1(G)$ and i, and thus the theorem is proved.

REMARK. If we refer to [4], we can see that $\sum_{\tau\in\mathfrak{T}(H)} L^1(H^\tau)$ is a subalgebra of $M(H)$ and that the norm closure of $\sum_{\tau\in\mathfrak{T}(H)} L^1(H^\tau)$ in $M(H)$ is a proper closed subalgebra of $M(H)$ if H is not discrete. This means that the set of the elements of the form $h(x)$ $(x\in G)$, where a LCA group G and a homomorphism h of $L^1(G)$ into $M(H)$ vary arbitrarily, constitutes the subalgebra $\sum_{\tau\in\mathfrak{T}(H)} L^1(H^\tau)$ contained (if H is not discrete) in a proper closed subalgebra of $M(H)$.

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