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ON INVERTIBLE OPERATORS AND INVARIANT SUBSPACES

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ABSTRACT. Let A be an invertible operator on a complex Hilbert space H. Sufficient conditions are given for the inverse of A to be a weak limit of polynomials in A.

1. **Introduction.** Let H be a complex Hilbert space. If H is finite dimensional and A is an invertible linear operator on H, then there is a polynomial p such that $A^{-1}=p(A)$. The infinite-dimensional analogue of this fact is generally false. If U is any unitary operator which contains a bilateral shift direct summand, then $U^{-1}=U^*$ is not a weak limit of polynomials in U [4]. In this paper two sufficient conditions, quite different in nature, are given for the inverse of a bounded linear operator to be a weak limit of polynomials in the operator.

2. **Preliminaries.** If A is a bounded linear operator on A, then Lat A represents the lattice of closed invariant subspaces of A. $H^{(n)}$ will denote the usual orthogonal direct sum of n copies of H. A typical vector in $H^{(n)}$ will be denoted by $\langle x_1, \cdots, x_n\rangle$ with $x_i \in H$. If A is an operator on H, let $A^{(n)}$ denote the operator $\sum_{i=1}^{n} \oplus A_i$ on $H^{(n)}$ with $A_i=A$ for all i. The inner product on H will be denoted by (,).

The following lemma is a special case of a well-known result [3].

LEMMA 1. *Let A be a bounded invertible operator on H. If* Lat $A^{(n)} \subseteq$ Lat $A^{-1(n)}$ *for all integers $n \geqq 1$, then A^{-1} is a weak limit of polynomials in A.*

3. **Numerical range.**

DEFINITION. Let A be a bounded operator on H. The numerical range of A is the set $\omega(A)=\{(Ax, x): \|x\|=1\}$.

LEMMA 2. *If $A \in B(H)$, then $\omega(A)=\omega(A^{(n)})$ for all integers $n \geqq 1$.*

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PROOF. Suppose $t \in \omega(A)$. Then there is a unit vector x in H such that $(Ax, x)=t$. A consideration of $(A^{(n)}y, y)$, for $y=\langle x, 0, \cdots, 0\rangle$, gives $t \in \omega(A^{(n)})$. Thus $\omega(A) \subset \omega(A^{(n)})$.

Now suppose t is in $\omega(A^{(n)})$. Thus there exist vectors $x_1, \cdots, x_n$ in H with $\sum_{i=1}^n \|x_i\|^2=1$ such that $\sum_{i=1}^n (Ax_i, x_i)=t$. Now $1/\|x_i\|^2(Ax_i, x_i) \in \omega(A)$ for $1 \leqq i \leqq n$. Since $\omega(A)$ is convex, it follows that

$$\sum_{i=1}^n \|x_i\|^2 \frac{1}{\|x_i\|^2}(Ax_i, x_i) = \sum_{i=1}^n (Ax_i, x_i) = t$$

is in $\omega(A)$. This completes the proof.

LEMMA 3. *Let A be a bounded invertible operator on H. Then* $0 \notin \omega(A)$ *implies* $\operatorname{Lat} A \subset \operatorname{Lat} A^{-1}$.

PROOF. Suppose there is $M \in \operatorname{Lat} A$ which is not invariant under A^{-1}. Since A is invertible, it follows that AM is a closed subspace of M. Let $N=M \ominus AM$.

Let x be a unit vector in N. $N \subset M$ implies $Ax \in AM$ and thus $(Ax, x)=0$. Since $0 \notin \omega(A)$, N must be zero.

NOTATION. The weak closure of the algebra of polynomials in A will be denoted by U_A.

THEOREM 1. *If A is an invertible operator on H and* $0 \notin \omega(A)$, *then* $U_A = U_{A^{-1}}$.

PROOF. $0 \notin \omega(A)$ implies $0 \notin \omega(A^{-1})$. For, if there exists $f \in H$ such that $(A^{-1}f, f)=0$, then since A is invertible, $f=Ag$, $g \in H$. Thus

$$0 = (A^{-1}f, f) = (A^{-1}Ag, Ag) = (g, Ag) = (A^*g, g).$$

By normalizing, if necessary, and using the fact that $\omega(A^*)=(\omega(A))^*$ we obtain $0 \in \omega(A)$.

Thus the result will be symmetric in A and A^{-1}. The fact that $\operatorname{Lat} A^{(n)} = \operatorname{Lat} A^{-1(n)}$ follows immediately from Lemmas 2 and 3. This completes the proof.

4. **Operator ranges.**

DEFINITION. A linear manifold $L \subset H$ is an operator range if there exists a Hilbert space K and a bounded operator A from K to H such that $L=AK$. The idea of studying the invariant operator ranges of an algebra of operators was introduced by Foias **[2]** and the basic facts about operator ranges can be found in an excellent account by Fillmore and Williams **[1]**.

If A is a bounded invertible operator on H, then a necessary condition for $A^{-1} \in U_A$ is $\operatorname{Lat} A \subset \operatorname{Lat} A^{-1}$. It is not known if this is also sufficient.

However, if every invariant linear manifold of A is invariant under A^{-1}, it follows from a result of P. Fillmore that $A^{-1}=p(A)$ for some polynomial p. Here we present what could be considered the intermediate result. The lattice of invariant operator ranges of A will be denoted by $\mathrm{Lat}_{1/2}\, A$.

THEOREM 2. *Let A be an invertible operator on H. Then* $\mathrm{Lat}_{1/2}\, A \subset \mathrm{Lat}_{1/2}\, A^{-1}$ *implies* $A^{-1} \in U_A$.

PROOF. We show, by induction, that $\mathrm{Lat}\, A^{(n)} \subseteq \mathrm{Lat}\, A^{-1(n)}$. By hypothesis, it is true for $n=1$ so assume $\mathrm{Lat}\, A^{(i)} \subset \mathrm{Lat}\, A^{-1(i)}$ for $i<n$ and suppose $M \in \mathrm{Lat}\, A^{(n)}$. We consider two cases.

Case (1). M does not contain a vector of the form $\langle 0, y_1, \cdots, y_{n-1}\rangle$ other than the zero vector. Then the first component uniquely determines every other component and, since M is a linear space, this determination is linear. Thus there exist (possibly unbounded) linear transformations $T_1, \cdots, T_{n-1}$ such that

$$M = \{\langle x, T_1x, \cdots, T_{n-1}x\rangle : x \in D\}$$

where

$$D = \{x : \exists x_1, \cdots, x_{n-1} \text{ with } \langle x, x_1, \cdots, x_{n-1}\rangle \in M\}.$$

Since M is closed and D is the range of the projection onto the first coordinate space of M, D is an operator range. $M \in \mathrm{Lat}\, A^{(n)}$ implies $D \in \mathrm{Lat}_{1/2}\, A \subset \mathrm{Lat}_{1/2}\, A^{-1}$.

Now $M \in \mathrm{Lat}\, A^{(n)}$ implies $AT_i=T_iA$ for $1 \leqq i \leqq n-1$. Thus

$$A^{-1}T_i = A^{-1}T_iAA^{-1} = A^{-1}AT_iA^{-1} = T_iA^{-1}.$$

Thus $M \in \mathrm{Lat}\, A^{-1(n)}$.

Case (2). Assume M contains a nontrivial vector $\langle 0, y_1, \cdots, y_{n-1}\rangle$. Let

$$N = \{\langle 0, y_1, \cdots, y_{n-1}\rangle \in M\}.$$

By the induction hypothesis $N \in \mathrm{Lat}\, A^{-1(n)}$:

Let $M'=M\ominus N$. The argument used in Case (1) shows that M' is of the form $\{\langle x, T_1x, \cdots, T_{n-1}x\rangle : x \in D\}$ with $D \in \mathrm{Lat}_{1/2}\, A \subset \mathrm{Lat}_{1/2}\, A^{-1}$. If $\langle x, T_1x, \cdots, T_{n-1}x\rangle \in M'$, then

$$A^{(n)}\langle x, T_1x, \cdots, T_{n-1}x\rangle = \langle Ax, T_1Ax, \cdots, T_{n-1}Ax\rangle + \langle 0, (AT_1 - T_1A)x, \cdots, (AT_{n-1} - T_{n-1}A)x\rangle,$$

where the first term is in M' and the second in N. Since $N \in \mathrm{Lat}\, A^{-1(n)}$,

$$A^{-1(n)}\langle 0, (AT_1 - T_1A)x, \cdots, (AT_{n-1} - T_{n-1}A)x\rangle = \langle 0, T_1x, \cdots, T_{n-1}x\rangle - \langle 0, A^{-1}T_1Ax, \cdots, A^{-1}T_{n-1}Ax\rangle$$

is in N. Let Q be the projection on $N^{\perp}$ in $H^{(n)}$. Then since $\langle 0, T_1x, \cdots, T_{n-1}x\rangle \in N^{\perp}$,

$$\langle 0, T_1x, \cdots, T_{n-1}x\rangle = Q\langle 0, A^{-1}T_1Ax, \cdots, A^{-1}T_{n-1}Ax\rangle.$$

We must show that $A^{-1(n)}\langle x, T_1x, \cdots, T_{n-1}x\rangle \in M$. Since $AD=D$, there is some $y \in D$ such that $x=Ay$. Thus $\langle x, T_1x, \cdots, T_{n-1}x\rangle = \langle Ay, T_1Ay, \cdots, T_{n-1}Ay\rangle$. Then

$$\begin{aligned} A^{-1(n)}\langle x, T_1x, \cdots, T_{n-1}x\rangle &= \langle y, A^{-1}T_1Ay, \cdots, A^{-1}T_{n-1}Ay\rangle \\ &= \langle y, 0, \cdots, 0\rangle + \langle 0, A^{-1}T_1Ay, \cdots, A^{-1}T_{n-1}Ay\rangle \\ &= \langle y, 0, \cdots, 0\rangle + Q\langle 0, A^{-1}T_1Ay, \cdots, A^{-1}T_{n-1}Ay\rangle \\ &\quad + (I^{(n)} - Q)\langle 0, A^{-1}T_1Ay, \cdots, A^{-1}T_{n-1}Ay\rangle \\ &= \langle y, 0, \cdots, 0\rangle + \langle 0, T_1y, \cdots, T_{n-1}y\rangle \\ &\quad + (I^{(n)} - Q)\langle 0, A^{-1}T_1Ay, \cdots, A^{-1}T_{n-1}Ay\rangle \\ &= \langle y, T_1y, \cdots, T_{n-1}y\rangle \\ &\quad + (I^{(n)} - Q)\langle 0, A^{-1}T_1Ay, \cdots, A^{-1}T_{n-1}Ay\rangle, \end{aligned}$$

which is clearly in M. This completes the proof.

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