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A SPACE OF SMALL SPREAD WITHOUT THE USUAL PROPERTIES

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ABSTRACT. A space is found, for any α, which has spread α and which is not the set-theoretic union of a hereditarily α-Lindelof and a hereditarily α-separable space.

Introduction. At the 1972 Bolyai János Mathematical Society Colloquium, A. Hajnal and I. Juhasz noted that every known Hausdorff space of spread ω was the union of a hereditary separable space and a hereditarily Lindelof space. The main result of this paper is a family of counterexamples to a generalization of this situation; the method of proof will also yield, in Lemma 2(c), a family of spaces such that no "large" subspaces are regular.

Some notational conventions. If X is a space, by its topology $\mathscr{T}$ we mean the family of open sets; if $\mathscr{A}$ is a family of subsets of X, the topology on X induced by $\mathscr{T}\cup\mathscr{A}$ is the closure of $\mathscr{T}\cup\mathscr{A}$ under arbitrary union and finite intersection. We write $\langle X,\mathscr{T}\rangle$ for X with the topology $\mathscr{T}$; if $Y\subset X$, $\langle Y,\mathscr{T}\rangle$ means $\langle Y,\{u\cap Y:u\in\mathscr{T}\}\rangle$. Given any set S, $|S|$ denotes the cardinality of S.

Statement of results.

DEFINITION. Given a topological space X, we define its spread by

$$\operatorname{sp}(X)=\sup\{|Y|: Y \text{ is a discrete subspace of } X\}.$$

DEFINITION. Let α be any cardinal, X a space. Then X is α-Lindelof iff every open cover of X has a subcover of cardinality $\leqq\alpha$. Similarly, X is α-separable iff every subspace has a dense set of cardinality $\leqq\alpha$.

DEFINITION. Let X be a space, P any property of topological spaces. Then X is hereditarily P iff every subspace of X has property P.

We note that if X is either hereditarily α-separable or hereditarily α-Lindelof, $\operatorname{sp}(X)\leqq\alpha$.

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THEOREM. *Let α be a cardinal. Then there exists a Hausdorff space X of cardinality α^+ such that* $\mathrm{sp}(X)=\alpha$ *and X is not the set-theoretic union of a hereditarily α-Lindelof space and a hereditarily α-separable space.*

COROLLARY OF PROOF. *For every cardinal α there exists a Hausdorff space of cardinality α^+ with no regular subspaces of cardinality α^+.*

Construction. From now on we fix some cardinal α. The construction proceeds by taking a space X of cardinality α^+ which is hereditarily α-separable and hereditarily α-Lindelof (any $X \subseteq 2^\alpha$, $|X|=\alpha^+$ will do). The points are then thought of as being indexed by the "square" array $\alpha^+ \times \alpha^+$. Lemma 1 ensures that no "vertical" or "diagonal" section is Lindelof; Lemma 2 ensures that no "horizontal" section is separable.

LEMMA 1. *Let X be a hereditarily α-separable space under the topology $\mathscr{T}$, and suppose X is the disjoint union of α^+ nonempty sets, $X=\bigcup_{\beta<\alpha^+} X_\beta$. Let $\mathscr{T}'$ be the topology induced on X by $\mathscr{T} \cup \{\bigcup_{\beta\leqq\gamma} X_\beta : \gamma<\alpha^+\}$. Then*

(a) *$\langle X, \mathscr{T}'\rangle$ is not α-Lindelof; in fact if $Y\subseteq X$, $|\{\beta : Y\cap X_\beta \neq \varnothing\}|=\alpha^+$ then Y is not α-Lindelof.*

(b) *$\langle X_\beta, \mathscr{T}'\rangle = \langle X_\beta, \mathscr{T}\rangle$ for all $\beta<\alpha^+$. Thus if X is hereditarily α-Lindelof under $\mathscr{T}$, $\langle X_\beta, \mathscr{T}'\rangle$ will be both hereditarily α-Lindelof and hereditarily α-separable.*

(c) *$\langle X, \mathscr{T}'\rangle$ is hereditarily α-separable.*

PROOF. (a) Let Y be as in the hypothesis, and consider the open cover of Y, $\{Y\cap \bigcup_{\beta\leqq\gamma} X_\beta : \gamma<\alpha^+\}$. Clearly no subfamily of cardinality α will cover Y.

(b) Clear.

(c) Let $Y\subseteq X$. Let A be a dense set of cardinality $\leqq\alpha$ for $\langle Y, \mathscr{T}\rangle$, and let $\gamma=\sup\{\beta : A\cap X_\beta\neq\varnothing\}$. If $y\in Y\cap\bigcup_{\beta\geqq\gamma} X_\beta$ and $y\in u\in\mathscr{T}'$ then $u\cap A\neq\varnothing$. For $\beta\leqq\gamma$, let A_β be dense for $\langle Y\cap X_\beta, \mathscr{T}'\rangle$, $|A_\beta|\leqq\alpha$. Then $A\cup\bigcup_{\beta\leqq\gamma} A_\beta$ is dense for $\langle Y, \mathscr{T}'\rangle$ and has cardinality $\leqq\alpha$.

LEMMA 2. *Let $X=\{x_\beta : \beta<\alpha^+\}$ be a hereditarily α-Lindelof space of cardinality α^+ with topology $\mathscr{T}$. Let $\mathscr{A}$ be any collection of subsets of X such that $|X-A|\leqq\alpha$ for all $A\in\mathscr{A}$. Let $\mathscr{T}'$ be the topology induced on X by $\mathscr{T}\cup\mathscr{A}$. Then*

(a) *$\langle X, \mathscr{T}'\rangle$ is hereditarily α-Lindelof.*

(b) *If, for all $\gamma<\alpha^+$, $\{x_\beta : \beta\geqq\gamma\}\in\mathscr{A}$, then $\langle X, \mathscr{T}'\rangle$ is not α-separable.*

(c) *If, for all $\gamma<\alpha^+$, $\{x_\beta : \beta\geqq\gamma\}\in\mathscr{A}$ and $\langle X, \mathscr{T}\rangle$ is hereditarily α-separable, then $\forall Y\subseteq X$ ($|Y|=\alpha^+ \rightarrow \langle Y, \mathscr{T}'\rangle$ is not regular).*

PROOF. (a) Let $Y\subseteq X$, $B\subset\mathscr{T}'$ be a basic open cover of Y. We may assume $\mathscr{A}$ is closed under finite intersection. Then $\forall b\in B$, $b=u\cap v$ for some $u\in\mathscr{T}$, $v\in\mathscr{A}$. Let $\mathscr{B}_{\mathscr{T}}=\{u\in\mathscr{T} : \exists b\in\mathscr{B}, \exists v\in\mathscr{A}\ (b=u\cap v)\}$, and let $\mathscr{C} \subseteq \mathscr{B}_{\mathscr{T}}$ be a subcover of Y, $|\mathscr{C}| \leqq \alpha$. Then $\forall u \in \mathscr{C}$, $\exists b \in \mathscr{B}$ such that $|u-b| \leqq \alpha$. For each $u \in \mathscr{C}$, fix such a $b \in \mathscr{B}$, calling it b_u, and let $\mathscr{C}_u \subset \mathscr{T}'$ cover $(u-b_u) \cap Y$, $|\mathscr{C}_u| \leqq \alpha$. Then $\{b_u : u \in \mathscr{B}_{\mathscr{T}}\} \cup \bigcup_{u \in \mathscr{B}_{\mathscr{T}}} \mathscr{C}_u$ is a subcover of Y in $\mathscr{T}'$ of cardinality α.

(b) Let $A \subseteq X$, $|A| \leqq \alpha$. Let $\gamma = \sup\{\beta : x_\beta \in A\}$. Then $\{x_\delta : \delta > \gamma\}$ is open and $A \cap \{x_\delta : \delta > \gamma\} = \varnothing$.

(c) Let $Y \subseteq X$, $|Y| = \alpha^+$. Since $\langle X, \mathscr{T}' \rangle$ is hereditarily α-Lindelof, we may without loss of generalization, assume that all open sets of $\langle Y, \mathscr{T}' \rangle$ have cardinality α^+. Suppose A is dense in $\langle Y, \mathscr{T} \rangle$, $|A| \leqq \alpha$. Again, let $\gamma = \sup\{\delta : x_\delta \in A\}$. Suppose $\beta > \alpha$. Then x_β is not an element of the closed set $\{x_\delta : \delta \leqq \gamma\} = w_\gamma$. We show that x_β and w_γ cannot be separated by open sets in $\mathscr{T}'$.

Let $u, v \in \mathscr{T}'$, $x_\beta \in u$, $w_\gamma \subset v$. Then $u = u' \cap a$, $v = v' \cap c$ for some u', $v' \in \mathscr{T}$, and $a, c \in \mathscr{A}$. Since A is dense relative to $\mathscr{T}$, $u' \cap v' \neq \varnothing$; hence $|u' \cap v'| = \alpha^+$. But then $|u \cap v| = |u' \cap v' \cap a \cap c| = \alpha^+$; clearly $u \cap v \neq \varnothing$.

Proposition. *There exists a Hausdorff space X of spread α such that if $X = Y_0 \cup Y_1$ then $\exists i \exists Z \exists Z'$ ($Z \subseteq Y_i$, $Z' \subseteq Y_i$, Z is not α-separable and Z' is not α-Lindelof).*

Proof. Let X be a hereditarily α-separable, hereditarily α-Lindelof Hausdorff space of spread α, $X = \bigcup_{\beta < \alpha^+} X_\beta$ as in Lemma 1, and suppose each X_β has cardinality α^+. Let $\mathscr{T}'$ be as in Lemma 1. We list the elements of X_β as $\{X_\delta^\beta : \delta < \alpha^+\}$ and note that $\langle X_\beta, \mathscr{T}' \rangle$ is hereditarily α-separable and hereditarily α-Lindelof. Let $\mathscr{A}_\beta$ be as in Lemma 2(b) for X_β. We construct the topology $\mathscr{T}^*$ as follows:

> Given $x_\delta^\beta \in X$, $u \in \mathscr{T}'$, $v \in \mathscr{A}_\beta$ such that $x_\delta^\beta \in u \cap v$, the following is a neighborhood basic open set: $u \cap [v \cup \bigcup_{\rho < \beta} X_\rho]$.

These sets are closed under finite intersection, hence they form a basis. Let $\mathscr{T}^*$ be the topology they generate. Clearly $\langle X, \mathscr{T}^* \rangle$ is Hausdorff and has spread $\geqq \alpha$. We show the spread is α: Suppose $Y \subseteq X$, $|Y| = \alpha^+$. Then either

(a) $\exists Z \subseteq Y$ such that $|\{\beta : Z \cap X_\beta \neq \varnothing\}| = \alpha^+$, or

(b) $\exists Z \subseteq Y$ such that $|Z| = \alpha^+$ and for some $\beta < \alpha^+$, $Z \subseteq X_\beta$.

In case (a) we may assume $|Z \cap X_\beta| \leqq 1$ for all $\beta < \alpha^+$. Then $\langle Z, \mathscr{T}^* \rangle = \langle Z, \mathscr{T}' \rangle$ and by Lemma 1, Z is hereditarily α-separable, hence not discrete. In case (b), by Lemma 2, Z is hereditarily α-Lindelof, hence not discrete. In either case, Y is not discrete. Now suppose $X = Y_0 \cup Y_1$. Suppose $|\{\beta : Y_0 \cap X_\beta \neq \varnothing\}| < \alpha^+$. Then letting $\gamma = \sup\{\beta : Y_0 \cap X_\beta \neq \varnothing\}$ we have $Y_1 \cap X_{\gamma+1}$ which is not α-separable, and $\{x_0^\delta : \delta > \gamma\}$ is a non-α-Lindelof subspace of Y_1. So we can assume $|\{\beta : Y_i \cap X_\beta \neq \varnothing\}| = \alpha^+$ for each i.

Hence neither Y_0 not Y_1 is α-Lindelof. Consider some $\delta<\alpha^+$. Then $|X_\delta \cap Y_{i_0}|=\alpha^+$ for some i_0. But then $X_\delta \cap Y_{i_0}$ is not α-separable, and this completes the proof.

In closing, we notice that by Lemma 2(c) this space is most definitely not regular; it would be interesting to know if a regular space can satisfy the main theorem.

References

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