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A TYCHONOFF ALMOST REALCOMPACTIFICATION

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ABSTRACT. Let X be a Tychonoff topological space. A Tychonoff almost realcompact space aX is constructed that contains X as a dense subspace and has the property that if $f: X \to Y$ is continuous and Y is Tychonoff and almost realcompact, then f can be extended continuously to aX. Several characterizations of aX are given, and the relationships between aX, the Hewitt realcompactification vX, and the minimal c-realcompactification uX are investigated. Properties of the projective covers of these spaces, and their relation to $vE(X)$ ($E(X)$ denotes the projective cover of X), are discussed.

1. **Introduction.** In [3], Frolik calls a Hausdorff topological space X *almost realcompact* if, given any ultrafilter $\mathscr{U}$ of open subsets of X such that $\bigcap_{n\in N} \mathrm{cl}_X U_n \neq \varnothing$ for each countable subfamily $(U_n)_{n\in N}$ of $\mathscr{U}$, then $\bigcap \{\mathrm{cl}_X U : U \in \mathscr{U}\} \neq \varnothing$. Frolik proved that the topological product of an arbitrary family of almost realcompact spaces is almost realcompact, and that a closed subspace of a regular almost realcompact space is almost realcompact. It follows from these results (and a theorem of Herrlich and van der Slot [7]) that corresponding to each Tychonoff space X there exists an almost realcompact space aX with the following properties: $X \subseteq aX \subseteq \beta X$ (the Stone-Čech compactification of X), and if f is a continuous function from X into any Tychonoff almost realcompact space Y, then f can be continuously extended to a function $f^a: aX \to Y$. In §2 we obtain a characterization of aX and discuss some of its properties. In §3 we discuss the relationship between aX and the c-realcompactification uX of X (see [2]), and consider the properties of the projective covers of these spaces. Finally, a comparison is made between aX and the Liu-Strecker almost realcompactification ρX that lies between X and its Katětov H-closed extension (see [8]). The notation and terminology of [4] are used throughout.

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A subset A of a topological space X is *regular closed* if $A=\mathrm{cl}_X(\mathrm{int}_X A)$. The family $\mathscr{R}(X)$ of all regular closed subsets of X is a complete Boolean algebra under the following operations:

(1) $A \leqq B$ iff $A \subseteq B$,

(2) $\bigvee_\alpha A_\alpha = \mathrm{cl}_X[\bigcup_\alpha A_\alpha]$,

(3) $\bigwedge_\alpha A_\alpha = \mathrm{cl}_X \mathrm{int}_X[\bigcap_\alpha A_\alpha]$,

(4) $A' = \mathrm{cl}_X(X-A)$.

It is immediate that Tychonoff almost realcompact spaces can be characterized as follows:

1.1 THEOREM. *A Tychonoff space X is almost realcompact if and only if each ultrafilter on $\mathscr{R}(X)$ with the countable intersection property (C.I.P.) has nonempty intersection.*

To conclude our introductory remarks, we briefly describe the projective cover, or absolute, of a Tychonoff space. A more detailed discussion may be found in [**11**]. Recall that a Hausdorff space is *extremally disconnected* if each of its open subsets has an open closure. If X is a Tychonoff space, then the Stone space $E(\beta X)$ of the Boolean algebra $\mathscr{R}(\beta X)$ is a compact extremally disconnected Hausdorff space whose points are ultrafilters on $\mathscr{R}(\beta X)$. If $A \in \mathscr{R}(\beta X)$, let $\lambda(A)=\{\alpha \in E(\beta X): A \in \alpha\}$; then the map $A \to \lambda(A)$ is a Boolean algebra isomorphism from $\mathscr{R}(\beta X)$ onto the Boolean algebra of open-and-closed subsets of $E(\beta X)$. Define $k: E(\beta X) \to \beta X$ as follows: if $\alpha \in E(\beta X)$, put $k(\alpha)=\bigcap \{A : A \in \alpha\}$. Then (see [**5**]) k is a well-defined continuous function from $E(\beta X)$ onto βX, k is *irreducible* (i.e. proper closed subsets of $E(\beta X)$ are mapped onto proper closed subsets of βX by k), and $k[\lambda(A)]=A$ for each $A \in \mathscr{R}(\beta X)$. Now $k^{\leftarrow}[X]$ is a dense, extremally disconnected, C^*-embedded subspace of $E(\beta X)$, and the restriction k_X of k to $k^{\leftarrow}[X]$ is a perfect irreducible map from $k^{\leftarrow}[X]$ onto X. The space $k^{\leftarrow}[X]$ can thus be identified with the projective cover $E(X)$ of X (in the category of Tychonoff spaces and perfect maps) discussed by Strauss in [**10**]. Evidently $E(\beta X)=\beta E(X)$, and if $X \subseteq T \subseteq \beta X$, then $k^{\leftarrow}[T]=E(T)$.

2. **The construction of** aX.

The following result of Herrlich and van der Slot appears as a corollary of Theorem 1 of [**7**].

2.1 THEOREM. *Let $\mathscr{P}$ be a topological property (of Tychonoff spaces) with the following properties:*

(a) *If each member of a family $\mathscr{F}$ of topological spaces has $\mathscr{P}$, then the product space $\prod \{F : F \in \mathscr{F}\}$ has $\mathscr{P}$.*

(b) *If X has $\mathscr{P}$ and S is a closed subspace of X, then S has $\mathscr{P}$.*

(c) *Compact spaces have $\mathscr{P}$.*

Then for each Tychonoff space X there exists a "maximal $\mathscr{P}$-extension" of X, denoted by γX, with the following properties:

(1) *γX is a Tychonoff space containing a dense copy of X.*

(2) *γX has $\mathscr{P}$.*

(3) *If Y is a Tychonoff space with $\mathscr{P}$ and if $f: X \to Y$ is continuous, then f can be continuously extended to a function $f^\gamma: \gamma X \to Y$.*

(4) *If $\bar{\gamma} X$ is another space satisfying* (1)–(3), *there is a homeomorphism from γX onto $\bar{\gamma} X$ that fixes X pointwise.*

(5) *γX can be identified with the intersection of all subspaces of βX that contain X and have $\mathscr{P}$.*

As an immediate consequence of the above theorem, we derive the following result.

2.2 THEOREM. *Corresponding to each Tychonoff space X there exists a Tychonoff almost realcompact space aX with the following properties*:

(1) *$X \subseteq aX \subseteq \beta X$.*

(2) *If f is a continuous function from X to a Tychonoff almost realcompact space Y, then f has a continuous extension f^a that maps aX into Y.*

In fact aX is the intersection of all the almost realcompact subspaces of βX that contain X, and f^a is the restriction to aX of the Stone extension f^β of f that maps βX into βY.

PROOF Let $\mathscr{P}$ be the topological property "almost realcompact and Tychonoff". Theorem 7 of **[3]** says that condition (a) of 2.1 is fulfilled for this $\mathscr{P}$, while Theorem 5 of **[3]** says that condition (b) of 2.1 is fulfilled. As each compact space obviously is almost realcompact, claims (1) and (2) above immediately follow from 2.1. Since $f^\beta | aX$ and f^a both map aX into βY and agree on X, they are equal. $\square$

Theorem 2.2 tells us that aX exists and that $X \subseteq aX \subseteq \beta X$; it does not tell us which points of $\beta X - X$ will be found in aX. As an analogy, the Hewitt realcompactification vX of X consists of those points p of βX such that the z-ultrafilter on X that converges to p has C.I.P. We wish to derive a similar characterization of aX. We begin with some technical lemmas. The first is a well-known result that follows easily from 8.7 of **[4]**.

2.3 THEOREM. *Let X be a Tychonoff space. Then $vX = \{p \in \beta X$: each G_δ-set of βX that contains p meets $X\}$.*

Let X be a Tychonoff space. An ultrafilter $\mathscr{A}$ on $\mathscr{R}(X)$ is said to *converge* to a point $p \in \beta X$ if $\{p\} = \bigcap \{\mathrm{cl}_{\beta X} A : A \in \mathscr{A}\}$. Evidently $\mathscr{A}$ converges to p if and only if $k(\alpha) = p$, where $k: E(\beta X) \to \beta X$ is the map defined in §1 and $\alpha = \{\mathrm{cl}_{\beta X} A : A \in \mathscr{A}\}$. Let $a_1 X$ denote the set $\{p \in \beta X$: there exists an ultrafilter $\mathscr{A}$ on $\mathscr{R}(X)$ with C.I.P. that converges to $p\}$. If n is a positive integer greater than 1, we define a_nX inductively as follows: $a_nX = a_1(a_{n-1}X)$.

We shall need the following result, which appears as 2.18 of [**12**].

2.4 LEMMA. *Let X be a Tychonoff space. Then*

$$vE(X) = \{\alpha \in E(\beta X) : \{A \cap X : A \in \alpha\} \text{ has C.I.P.}\}.$$

2.5 LEMMA. *Let X be a Tychonoff space. Let $k : E(\beta X) \to \beta X$ be the canonical map defined in* §1. *Then $k[vE(X)] = a_1X$.*

PROOF. Let $\alpha \in vE(X)$. Then $\{A \cap X : A \in \alpha\}$ has C.I.P. by 2.4, and converges to $k(\alpha)$. Hence $k(\alpha) \in a_1X$. Conversely, if $p \in a_1X$, find an ultrafilter $\mathscr{A}$ on $\mathscr{R}(X)$ such that $\mathscr{A}$ has C.I.P. and $\mathscr{A}$ converges to p. Put $\alpha = \{\mathrm{cl}_{\beta X} A : A \in \mathscr{A}\}$. Then $\mathscr{A} = \{A \cap X : A \in \alpha\}$, so $\alpha \in vE(X)$ by 2.4. Evidently $k(\alpha) = p$, so $p \in k[vE(X)]$. The lemma follows. $\square$

We shall need the following result which appears, among other places, as Theorem 1.7 of [**1**].

2.6 THEOREM. *The Tychonoff space X is almost realcompact if and only if $E(X)$ is realcompact.*

We need one more technical lemma, which perhaps is of independent interest.

2.7 LEMMA. *Let X be a Tychonoff space, and let $(T_n)_{n \in N}$ be a countable family of realcompact spaces such that $X \subseteq T_n \subseteq \beta X$ for each $n \in N$. Then $\bigcup_{n \in N} T_n$ is realcompact.*

PROOF. Put $Y = \bigcup_{n \in N} T_n$. As $X \subseteq Y \subseteq \beta X$, it follows that $\beta Y = \beta X$ (see 6.7 of [**4**]). Hence to show that Y is realcompact, it suffices to show that if $p \in \beta X - Y$, then there is a G_δ-set of βX containing p and disjoint from Y. But if $p \in \beta X - Y$, then $p \in \beta X - T_n$ for each $n \in N$, so as $\beta T_n = \beta X$ (since $X \subseteq T_n \subseteq \beta X$), there exists a G_δ-set G_n of βX such that $p \in G_n$ and $G_n \cap T_n = \varnothing$. Put $G = \bigcap_{n \in N} G_n$. Then G is a G_δ-set of βX containing p and disjoint from Y. Hence Y is realcompact. $\square$

2.8 THEOREM. *Let X be a Tychonoff space. Then $aX = \bigcup_{n \in N} a_nX$.*

PROOF. Repeated use of Lemma 2.5 shows that $k[vE(a_nX)] = a_{n+1}X$ and $vE(a_nX) \subseteq E(a_{n+1}X) \subseteq vE(a_{n+1}X)$ for each $n \in N$. Thus

$$E\left(\bigcup_{n \in N} a_nX\right) = k^{\leftarrow}\left[\bigcup_{n \in N} a_nX\right] = \bigcup_{n \in N} E(a_nX) = \bigcup_{n \in N} vE(a_nX).$$

By Lemma 2.7, $\bigcup_{n \in N} vE(a_nX)$ is realcompact. Hence by 2.6, $\bigcup_{n \in N} a_nX$ is almost realcompact. Hence $aX \subseteq \bigcup_{n \in N} a_nX$.

Each realcompact space is almost realcompact (see Theorem 10 of [3]), and each almost realcompact extremally disconnected space is realcompact (see Theorem 1.2 of [1]). Hence $aE(X)=vE(X)$. But the extension of the mapping $k_X:E(X)\to X$ to $aE(X)$ maps $aE(X)$ into aX (see 2.2); hence by 2.5 $a_1X\subseteq aX$. It then follows from 2.2 that $a(a_1X)=aX$. A repetition of this argument shows that $a_nX\subseteq aX$ for each $n\in N$. Hence $\bigcup_{n\in N} a_nX\subseteq aX$, and so $\bigcup_{n\in N} a_nX=aX$. $\square$

It is natural to conjecture that $aX=a_1X$ for any Tychonoff space X. I have been unable either to prove this conjecture or find a counterexample to it. More generally, it is possible that for each Tychonoff space X, $aX=a_nX$ for some positive integer n (perhaps depending on X). It is evident that if a_nX is almost realcompact for some $n\in N$, then $a_kX=a_nX$ for each $k\geqq n$.

We now consider some of the properties of aX. Note that since each realcompact space is almost realcompact, $aX\subseteq vX$ for each Tychonoff space X.

2.9 THEOREM. *Let X be a Tychonoff space. Then*:
(i) *vX is not locally compact at any part of $vX-a_1X$.*
(ii) *$\beta X-vX$ is dense in $\beta X-aX$.*
(iii) *If vX is locally compact then $a_1X=aX=vX$.*

PROOF. (1) In Theorem 2.8 of [12] it is shown that $E(\beta X)-E(vX)$ is dense in $E(\beta X)-vE(X)$. Applying the map k, and noting that $\beta X-a_1X\subseteq k[E(\beta X)-vE(X)]$, we conclude that $\beta X-vX$ is dense in $\beta X-a_1X$. Thus $\mathrm{cl}_{\beta X}(\beta X-vX)=\mathrm{cl}_{\beta X}(\beta X-a_1X)$. But $\beta X-\mathrm{cl}_{\beta X}(\beta X-vX)=\{p\in vX:vX$ is locally compact at $p\}$ (see 1.10 of [11]). Hence (i) follows. Statement (ii) now follows from the fact that $\beta X-vX\subseteq\beta X-aX\subseteq\beta X-a_1X$, and statement (iii) follows immediately from statement (i). $\square$

Some attention has been devoted to finding conditions on a pair of Tychonoff spaces X and Y that are equivalent to the truth of the equation $v(X\times Y)=vX\times vY$. A general solution of this problem has not been obtained. The following result relates this question to the corresponding question about almost realcompactifications.

2.10 THEOREM. *Let X and Y be Tychonoff spaces. If $v(X\times Y)=vX\times vY$, then $a(X\times Y)=aX\times aY$.*

PROOF. In general $X\times Y\subseteq a(X\times Y)\subseteq v(X\times Y)$. Since $vX\times vY=v(X\times Y)\subseteq\beta(X\times Y)$, it follows that $aX\times aY$ is an almost realcompact subspace of $\beta(X\times Y)$ that contains $X\times Y$. Hence $a(X\times Y)\subseteq aX\times aY$, so both $X\times aY$ and $a(X\times Y)$ are contained in $aX\times aY$. If $(p,q)\in X\times aY-a(X\times Y)$, then since $\{p\}\times aY$ and $a(X\times Y)$ are almost realcompact subspaces of the regular almost realcompact space $aX\times aY$, it follows that their intersection is almost realcompact (see Theorem 7 of [3]; note that regularity is needed). Thus $[\{p\}\times aY]\cap[a(X\times Y)]$ is almost realcompact and is properly contained in $\{p\}\times aY$ while containing $\{p\}\times Y$. This is a contradiction, so it follows that $X\times aY\subseteq a(X\times Y)$; evidently $a(X\times aY)=a(X\times Y)$. A repetition of this argument (with $aX\times aY$ in place of $X\times aY$) yields that $aX\times aY\subseteq a(X\times Y)$. Hence $aX\times aY=a(X\times Y)$. $\square$

3. **Almost realcompactifications and c-realcompactifications.** A Tychonoff space X is said to be *c-realcompact* if for each point $p\in\beta X-X$ there exists a normal lower semicontinuous function f on βX (see [9]) such that $f(p)=0$ and f is positive on X. Dykes defined c-realcompact spaces in [2]; they are discussed in some detail in [6]. The following result may be found in 1.1 and 2.5 of [6].

3.1 THEOREM. *Let X be a Tychonoff space. Let $uX=\{p\in\beta X$: each ultrafilter on $\mathscr{R}(X)$ that converges to p has C.I.P.$\}$. Then*:

(1) *uX is the smallest c-realcompact space between X and βX.*

(2) *X is c-realcompact if and only if given $p\in\beta X-X$, there exists a decreasing sequence $(A_n)_{n\in N}\subseteq\mathscr{R}(\beta X)$ such that $p\in\bigcap_{n\in N}A_n$ and $\bigcap_{n\in N}A_n\cap X=\varnothing$.*

The space uX is called the c-realcompactification of X.

It is proved in 3.3 of [2] that each almost realcompact space is c-realcompact. Hence if X is a Tychonoff space, then $X\subseteq uX\subseteq aX\subseteq vX$. The relationship between uX and aX is clarified in the following.

3.2 LEMMA. *Let X be a Tychonoff space. Then*:

(a) *aX is the smallest space T between X and βX such that $E(T)$ is realcompact. In particular, $vE(X)\subseteq E(aX)$.*

(b) *uX is the largest space T between X and βX such that $E(T)\subseteq vE(X)$.*

PROOF. Recall that $X\subseteq T\subseteq\beta X$ iff $E(X)\subseteq E(T)\subseteq E(\beta X)=\beta E(X)$. Part (a) now follows from 2.2 and 2.6.

Let $\alpha\in E(uX)$. Then α is an ultrafilter on $\mathscr{R}(\beta X)$ such that $\{A\cap X: A\in\alpha\}$ converges to $k(\alpha)\in uX$. Thus $\{A\cap X: A\in\alpha\}$ has C.I.P., and so $\alpha\in vE(X)$ by 2.4. Hence $E(uX)\subseteq vE(X)$. Conversely, if $p\in\beta X-uX$, there is an ultrafilter α on $\mathscr{R}(\beta X)$ such that $\{A\cap X: A\in\alpha\}$ converges to p but does not have C.I.P. Hence $\alpha\notin vE(X)$ so $k^{\leftarrow}(p)-vE(X)\neq\varnothing$. Hence (b) holds.

3.3 THEOREM. *The following conditions on a Tychonoff space X are equivalent*:

(a) *$vE(X)=E(T)$ for some T such that $X\subseteq T\subseteq\beta X$.*

(b) *$vE(X)=E(aX)$.*

(c) *$uX=aX$.*

(d) *$uX=a_1X$.*

PROOF. (a)$\Rightarrow$(b): If $vE(X)=E(T)$ for some T such that $X\subseteq T\subseteq \beta X$, then $E(T)$ is realcompact so T is almost realcompact. Hence $aX\subseteq T$. But by 3.2(a), $vE(X)\subseteq E(aX)$, so $T\subseteq aX$. Hence $T=aX$.

(b)$\Rightarrow$(c): Evidently $E(aX)\subseteq vE(X)$ so, by 3.2(b), $aX\subseteq uX$. Hence $aX=uX$.

(c)$\Rightarrow$(d): In general $uX\subseteq a_1X\subseteq aX$, so if $uX=aX$ then $uX=a_1X$.

(d)$\Rightarrow$(a): By 2.5, $k[vE(X)]=a_1X$ so $vE(X)\subseteq E(a_1X)$. Thus $E(uX)\subseteq vE(X)\subseteq E(a_1X)=E(uX)$ so $vE(X)=E(uX)$. $\square$

An example of a Tychonoff space X such that $uX\neq aX$ can be found in the example on pp. 240–241 of [**9**]. It is constructed as follows: let T be the Tychonoff plank (see 8.20 of [**4**]), let $A=W^*\times\{\omega\}$ and $B=\{\omega_1\}\times N^*$ denote, respectively, the top and right edge of T^*. Let X^* denote the space obtained from $T^*\times N$ by identifying $A\times\{2n-1\}$ with $A\times\{2n\}$ and identifying $B\times\{2n\}$ with $B\times\{2n+1\}$. Let t denote the corner point (ω_1, ω, n) of X^*, and put $X=X^*-\{t\}$. Mack and Johnson show in [**9**] that $X^*=vX$ and that X is not weak *cb* (see §3 of [**9**]) while X^* is weak *cb*. In Theorem 1.11 of [**6**] it is shown that X is weak *cb* iff uX is weak *cb*. Hence $X=uX$, i.e. X is c-realcompact.

Let $f: T^*\times N\to X^*$ denote the identification map described above. Then f takes $T\times N$ onto X. Obviously $T^*\times N=v(T\times N)$, so by 2.9(iii), $a(T\times N)=T^*\times N$. Hence the extension of $f|T\times N$ to $T^*\times N$ maps $T^*\times N$ into aX. But $f[T^*\times N]=X^*=vX$, so $aX=vX\neq uX$.

We conclude this paper by comparing the Strecker-Liu almost realcompactification ρX of X (see [**8**]) to aX and uX. Recall that the *Katětov H-closed extension* κX of the Hausdorff space X is formed as follows: Let Y be the family of all open ultrafilters $\mathscr{U}$ on X such that

$$\bigcap\,\{\mathrm{cl}_X\, U: U\in\mathscr{U}\} = \varnothing,$$

and let $\kappa X=X\cup Y$, topologized as follows: open subsets of X remain open in κX, and if $p\in Y$ then $\{\{p\}\cup G: G\in p\}$ is a neighbourhood base at p in κX. Then κX is a Hausdorff space that is a closed subspace of each Hausdorff space in which it can be embedded (i.e. κX is H-closed), and if hX is any other H-closed Hausdorff space that contains X as a dense subspace, then there is a continuous map from κX onto hX that fixes X pointwise. Let $\rho X=X\cup\{p\in Y: \bigcap_{n\in N}\mathrm{cl}_X\, G_n\neq\varnothing$ for each countable subfamily $(G_n)_{n\in N}$ of $p\}$. Liu and Strecker prove that ρX (endowed with the subspace topology inherited from κX) is an almost realcompact Hausdorff space with the property that if S is an an almost realcompact Hausdorff space that contains X as a dense subspace, then ρX can be mapped into S by a continuous function that fixes X pointwise. Thus ρX plays the same role in the class of Hausdorff spaces as aX plays in the class of Tychonoff spaces.

The relation between ρX, $a_1 X$, and aX is described in the following theorem.

3.4 THEOREM. *Let X be a Tychonoff space and let f: $\kappa X \to \beta X$ be the unique continuous function that is the identity on X. Then*:

(1) $f[\rho X] = a_1 X$.

(2) $\rho X = f^{\leftarrow} f[\rho X]$ *if and only if* $uX = a_1 X$.

PROOF. (1) It is easily seen that, if $p \in \kappa X - X$, then

$$f(p) = \bigcap \{\mathrm{cl}_{\beta X} U : U \in p\}.$$

If $p \in \rho X - X$, then $\{\mathrm{cl}_X U : U \in \mathscr{U}\}$ is an ultrafilter on $\mathscr{R}(X)$ with C.I.P. that converges to $f(p) \in \beta X$. Thus $f(p) \in a_1 X$. Conversely, if $q \in a_1 X$, then find an ultrafilter $\mathscr{A}$ on $\mathscr{R}(X)$ with C.I.P. such that $\mathscr{A}$ converges to q. Let $p = \{V : V$ is open in X and $\mathrm{int}_X A \subseteq V$ for some $A \in \mathscr{A}\}$. Then $p \in \rho X$ and $f(p) = q$. Hence $f[\rho X] = a_1 X$.

(2) Assume that $\rho X = f^{\leftarrow}[f[\rho X]] = f^{\leftarrow}[a_1 X]$. Let $q \in a_1 X$ and let $\mathscr{A}$ be any ultrafilter on $\mathscr{R}(X)$ that converges to q. Let $p = \{V : V$ is open in X and $\mathrm{int}_X A \subseteq V$ for some $A \in \mathscr{A}\}$. Then $p \in \kappa X$ and $f(p) = q$. Thus $p \in f^{\leftarrow}[a_1 X] = \rho X$. Hence $\mathscr{A}$ has C.I.P., and so each ultrafilter on $\mathscr{R}(X)$ that converges to q has C.I.P. Hence $q \in uX$ and so $a_1 X \subseteq uX$. But $uX \subseteq a_1 X$ in general, so $uX = a_1 X$. Conversely, if $uX = a_1 X$, let $p \in f^{\leftarrow}[a_1 X] = f^{\leftarrow}[uX]$. Then the ultrafilter $\{\mathrm{cl}_X V : V \in p\}$ on $\mathscr{R}(X)$ converges to $f(p) \in uX$. Thus this ultrafilter has C.I.P., and so $p \in \rho X$. Thus $f^{\leftarrow} f[\rho X] = \rho X$. $\square$

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