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A PRODUCT VARIETY OF GROUPS WITH DISTRIBUTIVE LATTICE

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ABSTRACT. By a *variety of A-groups* is meant a locally finite variety of groups whose nilpotent groups are abelian. It is shown that if $\mathfrak{U}$ is a variety of A-groups and $\mathfrak{V}$ is a locally finite variety whose lattice of subvarieties is distributive and the exponents of $\mathfrak{U}$ and $\mathfrak{V}$ are coprime, then the lattice of subvarieties of the product variety $\mathfrak{U}\mathfrak{V}$ is distributive.

1. **Introduction.** The *lattice of a variety* $\mathfrak{V}$ *of groups* is the lattice of subvarieties of $\mathfrak{V}$ partially ordered by inclusion. It is modular because the lattice of the variety of all groups is dual to the lattice of fully invariant subgroups of the free group of countably infinite rank. For any positive integer m let $\mathfrak{A}_m$, $\mathfrak{B}_m$ and $\mathfrak{N}_m$ denote respectively the variety of all abelian groups of exponent dividing m, the variety of all groups of exponent dividing m, and the variety of all groups which are nilpotent of class at most m. A *variety of A-groups* is defined to be a locally finite variety whose nilpotent groups are abelian. G. Higman [7, 54.24] gave the first example of a variety with a nondistributive lattice. R. A. Bryce [3, 6.2.5] showed that for a prime p the product variety $\mathfrak{A}_{p^2}\mathfrak{A}_{p^2}$ has a nondistributive lattice but that a variety of metabelian groups of bounded exponent in which, for each p, the p-groups have class at most p has distributive lattice. He also showed that if m is nearly prime to n (*i.e.* if a prime p divides m then p^2 does not divide n) then the lattice of $\mathfrak{A}_m\mathfrak{A}_n$ is distributive. M. S. Brooks [1] showed that the lattice of $\mathfrak{A}_3\mathfrak{A}_9$ is not distributive. The main result here generalizes one of John Cossey [4] who showed that the lattice of varieties of A-groups is distributive. The *exponent* of a locally finite variety is defined to be the order of the free group on one generator of the variety.

THEOREM 1. *Suppose* $\mathfrak{U}$ *is a variety of A-groups and* $\mathfrak{V}$ *is a locally finite variety with distributive lattice and the exponents of* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are coprime. Then the lattice of* $\mathfrak{U}\mathfrak{V}$ *is distributive.*

Notation and terminology not here defined are as in Hanna Neumann [7]. In view of Theorem 1 it is worth noting that L. G. Kovács has an

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unpublished example which shows that although the lattice of the meet $\mathfrak{B}_8 \wedge \mathfrak{N}_3$ is distributive, that of $(\mathfrak{B}_8 \wedge \mathfrak{N}_3)\mathfrak{A}_3$ is not.

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2. **A theorem on skeletons.** By a *section* of a group is meant a factor group of a subgroup of it. If $\mathscr{G}$ is a class of groups then $\mathrm{s}\mathscr{G}$ and $\mathrm{Q}\mathscr{G}$ denote the classes of all groups isomorphic to, respectively, subgroups and factor groups of groups in $\mathscr{G}$. A class $\mathscr{G}$ of groups is said to be *section closed* if $\mathrm{Q}\mathscr{G} \subseteq \mathscr{G}$ and $\mathrm{s}\mathscr{G} \subseteq \mathscr{G}$. It is well known and easy to see that if $\mathscr{G}$ is a class of groups then $\mathrm{QS}\mathscr{G}$ is section closed. The *skeleton* $\mathscr{S}(\mathfrak{B})$ *of a variety* $\mathfrak{B}$ is defined (in Bryant and Kovács [**2**]) to be the intersection of the section closed classes of groups generating $\mathfrak{B}$. A *monolithic group* is defined to be a finite group with a unique minimal normal subgroup, called the *monolith*. To prove Theorem 1 we need the following result.

THEOREM 2. *Suppose p is a prime and $\mathfrak{Y}$ is a locally finite variety containing a variety $\mathfrak{X}$ of p'-exponent such that for some positive integer α, $\mathfrak{Y}$ is contained in $\mathfrak{A}_{p^\alpha}\mathfrak{X}$,*

$$\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{A}_{p^\alpha}\mathfrak{X},$$

and $\mathfrak{Y}$ is generated by monolithic groups not in $\mathfrak{X}$. Then

$$\mathscr{S}(\mathfrak{Y}) = \mathrm{QS}\{G \mid G \in \mathfrak{Y}, G \notin \mathfrak{X} \text{ and } G \text{ is monolithic}\},$$

and $\mathscr{S}(\mathfrak{Y})$ generates $\mathfrak{Y}$.

PROOF. Let G be a monolithic group in $\mathfrak{Y}$ but not in $\mathfrak{X}$, let σG be the monolith of G, σ^*G be the centralizer of σG in G, $Z(G)$ be the center of G, $X = \mathfrak{X}(G)$ be the $\mathfrak{X}$-verbal subgroup of G, and G' be the derived group of G. We write $H \lhd G$ if H is a normal subgroup of G.

Notice X is the Sylow p-subgroup of σ^*G; we show they are equal. If σ^*G is not abelian then

$$\sigma G \leqq (\sigma^*G)' \cap Z(\sigma^*G) \cap X = 1$$

by [**6**, IV2.2], which is a contradiction. Thus σ^*G is abelian and, since G is monolithic, σ^*G is of prime power order. Because

$$\sigma G \leqq X \leqq \sigma^*G,$$

we have $X = \sigma^*G$.

Let $\mathscr{H}$ be a section closed class of groups generating $\mathfrak{Y}$. To prove the theorem it suffices to show $G \in \mathscr{H}$. We shall use some properties of the minimal representation defined in [7, p. 163 ff]. Let

$$G \cong H/K, \qquad H \leqq H_1 \times \cdots \times H_r, H_i \in \mathscr{H} \text{ for } i = 1, \cdots, r$$

be a minimal representation of G on $\mathscr{H}$. Then each H_i is monolithic and $\sigma H_i \cong \sigma G$ so σH_i is a p-group. By the last paragraph $\sigma^* H_i = \mathfrak{X}(H_i)$. By the Schur-Zassenhaus theorem there is a complement, K_i say, for $\sigma^* H_i$ in H_i. Since H_i is monolithic, $\sigma^* H_i$ is an indecomposable K_i-group so by [**5**, 5.2.2], $\sigma^* H_i$ is a homocyclic p-group. For some j the exponent of $\sigma^* H_j$ is greater than or equal to the exponent of $\sigma^* G$. Let n be the exponent of $\sigma^* G$. It follows as in Lemma 3 of Cossey [**4**] that $G \cong H_j/(\sigma^* H_j)^n$, and $G \in \mathscr{H}$, proving the theorem.

3. **Proof of Theorem 1.** Let $\mathfrak{U}_1, \mathfrak{U}_2 \leqq \mathfrak{U}\mathfrak{B}$. We first show

$$(*) \qquad (\mathfrak{B} \wedge \mathfrak{U}_1) \vee (\mathfrak{B} \wedge \mathfrak{U}_2) = \mathfrak{B} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2).$$

Since $(\mathfrak{B} \wedge \mathfrak{U}_1) \subseteq \mathfrak{B} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2)$ it suffices to prove that if F is a finite free group of $\mathfrak{B} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2)$ then $F \in (\mathfrak{B} \wedge \mathfrak{U}_1) \vee (\mathfrak{B} \wedge \mathfrak{U}_2)$. Let $\mathfrak{U}_1 \cup \mathfrak{U}_2$ denote the set theoretic union of $\mathfrak{U}_1$ and $\mathfrak{U}_2$. Let

$$F \cong H/K, \qquad H \leqq H_1 \times \cdots \times H_r, H_i \in (\mathfrak{U}_1 \cup \mathfrak{U}_2) \text{ for } i = 1, \cdots, r$$

be a minimal representation of F on $\mathfrak{U}_1 \cup \mathfrak{U}_2$. Because $F \in \mathfrak{B}$, σH_i has exponent dividing that of $\mathfrak{B}$. Since the exponents of $\mathfrak{U}$ and $\mathfrak{B}$ are relatively prime it follows that $H_i \in \mathfrak{B}$ for all i. As $H_i \in \mathfrak{U}_1 \cup \mathfrak{U}_2$ we have $H_i \in (\mathfrak{B} \cap \mathfrak{U}_1) \cup (\mathfrak{B} \cap \mathfrak{U}_2)$. It follows that $F \in (\mathfrak{B} \wedge \mathfrak{U}_1) \vee (\mathfrak{B} \wedge \mathfrak{U}_2)$, proving (*).

We need a lemma.

LEMMA. *If G is a monolithic group in $\mathfrak{U}_1 \vee \mathfrak{U}_2$ but not in $\mathfrak{B}$ then $G \in \mathfrak{U}_1 \cup \mathfrak{U}_2$.*

PROOF. If σG is not abelian then by taking a minimal representation of G on $\mathfrak{U}_1 \cup \mathfrak{U}_2$ and arguing as in [7, 53.31] the result follows. Thus we may assume σG is an abelian p-group for some prime p. Let

$$G \cong H/K, \qquad H \leqq H_1 \times \cdots \times H_r, H_i \in (\mathfrak{U}_1 \cup \mathfrak{U}_2) \text{ for } i = 1, \cdots, r$$

be a minimal representation of G on $\mathfrak{U}_1 \cup \mathfrak{U}_2$. Let $V_i = (\sigma^* H_i) \cap \mathfrak{B}(H_i)$ and observe that the Sylow p-subgroups of H_i are in V_i and $\sigma H_i \leqq Z(V_i)$. As V_i is an A-group, $Z(V_i) \cap V_i' = 1$. Since H_i is monolithic, $V_i' = 1$. Thus V_i is abelian and must be a p-group.

Let $\mathfrak{Y}$ be the variety generated by $H_1, \cdots, H_r$ and $\mathfrak{X}$ be the variety generated by $H_1/V_1, \cdots, H_r/V_r$. Then by Theorem 2,

$$\mathscr{S}(\mathfrak{Y}) = \mathrm{QS}\{H \mid H \in \mathfrak{Y}, H \notin \mathfrak{X} \text{ and } H \text{ is monolithic}\}.$$

It follows that

$$G \in \mathscr{S}(\mathfrak{Y}) \subseteq \mathrm{QS}\{H_1, \cdots, H_r\} \subseteq \mathfrak{U}_1 \cup \mathfrak{U}_2$$

proving the lemma.

To prove Theorem 1 it suffices to show that if $\mathfrak{W} \subseteq \mathfrak{U}\mathfrak{V}$ then

$$\mathfrak{W} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2) = (\mathfrak{W} \wedge \mathfrak{U}_1) \vee (\mathfrak{W} \wedge \mathfrak{U}_2).$$

Since $\mathfrak{W} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2) \supseteq \mathfrak{W} \wedge \mathfrak{U}_1$ it suffices to show that if G is a monolithic group in $\mathfrak{W} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2)$ then G is in $(\mathfrak{W} \wedge \mathfrak{U}_1) \vee (\mathfrak{W} \wedge \mathfrak{U}_2)$.

Suppose first that $G \notin \mathfrak{V}$. Then by the lemma $G \in \mathfrak{U}_1 \cup \mathfrak{U}_2$. As $G \in \mathfrak{W}$,

$$\begin{aligned} G \in \mathfrak{W} \cap (\mathfrak{U}_1 \cup \mathfrak{U}_2) &= (\mathfrak{W} \cap \mathfrak{U}_1) \cup (\mathfrak{W} \cap \mathfrak{U}_2) \\ &\subseteq (\mathfrak{W} \wedge \mathfrak{U}_1) \vee (\mathfrak{W} \wedge \mathfrak{U}_2). \end{aligned}$$

Suppose $G \in \mathfrak{V}$. Using the fact that $\mathfrak{V}$ has distributive lattice and applying (*) twice, we have

$$\begin{aligned} \mathfrak{V} \wedge \mathfrak{W} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2) &= \mathfrak{V} \wedge \mathfrak{W} \wedge [\mathfrak{V} \wedge (\mathfrak{U}_1 \vee \mathfrak{U}_2)] \\ &= (\mathfrak{V} \wedge \mathfrak{W}) \wedge [(\mathfrak{V} \wedge \mathfrak{U}_1) \vee (\mathfrak{V} \wedge \mathfrak{U}_2)] \\ &= [(\mathfrak{V} \wedge \mathfrak{W}) \wedge (\mathfrak{V} \wedge \mathfrak{U}_1)] \vee [(\mathfrak{V} \wedge \mathfrak{W}) \wedge (\mathfrak{V} \wedge \mathfrak{U}_2)] \\ &= (\mathfrak{V} \wedge \mathfrak{W} \wedge \mathfrak{U}_1) \vee (\mathfrak{V} \wedge \mathfrak{W} \wedge \mathfrak{U}_2) \\ &= \mathfrak{V} \wedge [(\mathfrak{W} \wedge \mathfrak{U}_1) \vee (\mathfrak{W} \wedge \mathfrak{U}_2)]. \end{aligned}$$

This completes the proof of the theorem.

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