

AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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ABSTRACT. Let $\mathcal{P}_{n,b}$ denote the class of all polynomials $p_n(z)$ of degree at most n in z which satisfy $\max_{|z|=1} |p_n(z)| = 1$, and $|p_n(1)| = b$, $0 \leq b < 1$. Let $c \in (0, n]$, and set

$$\mu_b(c, n) = \sup_{p_n \in \mathcal{P}_{n,b}} \left\{ \min_{|z|=1-c/n} |p_n(z)| \right\}.$$

Upper estimates for $\mu_b(c, n)$ are obtained.

Let U denote the open unit disc in the complex z plane, T its boundary, and let $\mathcal{P}_{n,0}$ denote the class of all polynomials $p_n(z)$ of degree at most n in z , satisfying $\max_{z \in T} |p_n(z)| = 1$ and $p_n(1) = 0$. The extremal problem in question is to estimate

$$\mu(c, n) = \sup_{p_n \in \mathcal{P}_{n,0}} \left\{ \min_{|z|=1-c/n} |p_n(z)| \right\},$$

where $0 < c \leq n$. This problem was mentioned by Professor Paul Erdős during a lecture at the University of Montreal in July, 1971. He attributed the problem to G. Halász, of the Mathematical Institute of the Hungarian Academy of Sciences; Erdős asked if there exists a constant c such that $\mu(c, n) = 1 - \varepsilon_n$ where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

It is easily seen that no such constant c exists. In fact, if $p_n \in \mathcal{P}_{n,0}$, then also $q_n \in \mathcal{P}_{n,0}$, where $q_n(z) = z^n p_n(1/z)$, and by S. Bernstein's theorem [3, p. 45] on the derivative of a polynomial, $|q'_n(z)| \leq n$ for $z \in T$. Hence it follows that $|z^{n-1} q'_n(1/z)| \leq n$ for $z \in T$ and by the maximum principle, also for all $z \in U$. Replacing z by $1/z$ we find that

$$|q'_n(z)| \leq n|z|^{n-1} \quad \text{for all } |z| \geq 1.$$

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Consequently,

$$|q_n((1 - c/n)^{-1})| = \left| \int_1^{(1-c/n)^{-1}} q'_n(t) dt \right|$$

$$\leq (1 - c/n)^{-n} - 1 = (1 - c/n)^{-n} \{1 - (1 - c/n)^n\},$$

that is,

$$(1) \quad |p_n(1 - c/n)| = |(1 - c/n)^n q_n((1 - c/n)^{-1})|$$

$$\leq 1 - (1 - c/n)^n \rightarrow 1 - e^{-c} \quad \text{as } n \rightarrow \infty.$$

The inequality (1) provides a negative answer to the question raised by Erdős and also gives an upper estimate for $\mu(c, n)$. However, this estimate is quite crude. The following theorem, which we shall prove, gives “essentially” best possible upper estimates for $\mu(c, n)$.

THEOREM 1. *In the above notation,*

$$(2) \quad \mu(c, n) < \{1 - (1 - c/n)^n\} / \{1 + (1 - c/n)^n\} \quad \text{if } 0 < c \leq 1,$$

and

$$(3) \quad \mu(c, n) < \frac{\{(2n - 1)c - (2n - c)(1 - 1/n)^n\}}{\{(2n - 1)c + (2n - c)(1 - 1/n)^n\}} \quad \text{if } 1 < c \leq n.$$

The right-hand side of (2) is equal to $c/2 + o(c)$ as $c \rightarrow 0$; moreover, the polynomial $p_n(z) = (1 - z^n)/2$ satisfies

$$\min_{|z|=1-c/n} |p_n(z)| = |p_n(1 - c/n)| = \{1 - (1 - c/n)^n\} / 2 = c/2 + o(c)$$

as $c \rightarrow 0$. Consequently, the inequality (2) is the best possible in the limit as $c \rightarrow 0$.

We find from (3) that $\mu(c, n) \leq 1 - 1/ec + o(1/c)$ as $c \rightarrow \infty$. We shall show that the function $1/(ec)$ cannot be replaced by one which approaches zero more slowly with regards to order, as $c \rightarrow \infty$. We prove

THEOREM 2. *Given*

$$\lambda > \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \log \left(1 - \frac{\sin^2 u}{u^2} \right) \right| du$$

there exists a positive number $A(\lambda)$, depending only on λ , such that whenever $c > A(\lambda)$, then

$$\mu(c, n) > \exp(-\lambda/c) > 1 - \lambda/c.$$

For the proof of Theorem 1 we use two subsidiary results.

LEMMA 1 [1, THEOREM 4]. *Let D be a circular domain in the z -plane, and S an arbitrary set of points in the w -plane. If the polynomial p_n of degree*

n satisfies $p_n(z) = w \in S$ for all $z \in D$, then for all $z \in D$ and all $\zeta \in D$,

$$\frac{\zeta p'_n(z)}{n} + p_n(z) - \frac{z p'_n(z)}{n} \in S.$$

LEMMA 2. If $f(z)$ is analytic in U , where it satisfies $|f(z)| \leq 1$, then for $0 \leq \alpha < 2\pi$ and $0 \leq r_1 < r_2 < 1$,

$$(4) \quad f(r_1 e^{i\alpha}) \leq (A - B)/(A + B)$$

where

$$A = (1 + r_2)(1 - r_1)\{1 + |f(r_2 e^{i\alpha})|\},$$

$$B = (1 - r_2)(1 + r_1)\{1 - |f(r_2 e^{i\alpha})|\}.$$

PROOF OF LEMMA 2. It is well known that if $f(z)$ is analytic in U , where it satisfies $|f(z)| \leq 1$, then

$$|f'(z)|/(1 - |f(z)|^2) \leq 1/(1 - |z|^2) \quad \text{for all } z \in U.$$

Hence

$$\left| \int_{r_1}^{r_2} \frac{(d/dr) |f(re^{i\alpha})|}{1 - |f(re^{i\alpha})|^2} dr \right| \leq \int_{r_1}^{r_2} \frac{|f'(re^{i\alpha})|}{1 - |f(re^{i\alpha})|^2} dr \leq \int_{r_1}^{r_2} \frac{dr}{1 - r^2}.$$

Now if $|f(r_1 e^{i\alpha})| > |f(r_2 e^{i\alpha})|$, we get

$$\left\{ \frac{1 + G_1}{1 - G_1} \right\} / \left\{ \frac{1 + G_2}{1 - G_2} \right\} \leq \left\{ \frac{1 + r_2}{1 - r_2} \right\} / \left\{ \frac{1 + r_1}{1 - r_1} \right\}$$

where $G_k = |f(r_k e^{i\alpha})|$, $k=1, 2$, which readily gives the desired estimate of $|f(r_1 e^{i\alpha})|$. The inequality (4) is trivially true if $|f(r_1 e^{i\alpha})| \leq |f(r_2 e^{i\alpha})|$.

PROOF OF THEOREM 1. Let $p_n \in \mathcal{P}_{n,0}$, $0 < c \leq 1$, and let

$$\min_{|z|=1-c/n} |p_n(z)| = a.$$

We wish to show that

$$a < \{1 - (1 - c/n)^n\} / \{1 + (1 - c/n)^n\}.$$

Without loss of generality we may suppose that $p_n(z) \neq 0$ in U , and therefore

$$\min_{|z| \leq 1-c/n} |p_n(z)| = \min_{|z|=1-c/n} |p_n(z)| = a.$$

This implies that p_n maps the circular domain $D = \{z : |z| \leq 1 - c/n\}$ onto a set S which lies in the ring $\{w : a \leq |w| < 1\}$. Hence by Lemma 1,

$$(1 - c/n) |p'_n(z)|/n < (1 - a)/2$$

for all $|z|=1-c/n$, i.e.,

$$|p'_n((1-c/n)z)| < \frac{1}{2}(1-a)n^2/(n-c) \quad \text{for all } |z|=1.$$

The same inequality holds for the polynomial $z^{n-1}p'_n((1-c/n)/z)$. Using the maximum modulus principle, we therefore conclude that

$$|z^{n-1}p'_n((1-c/n)/z)| \leq \frac{1}{2}(1-a)n^2/(n-c) \quad \text{for all } z \in (U \cup T).$$

Replacing z by $(1-c/n)/z$ we obtain

$$|p'_n(z)| < \frac{1}{2}\{(1-a)n^2/(n-c)\}\{z/(1-c/n)\}^{n-1} \quad \text{for all } |z| \geq 1-c/n.$$

This implies that

$$\begin{aligned} 0 &= |p_n(1)| = \left| p_n(1-c/n) + \int_{1-c/n}^1 p'_n(t) dt \right| \\ &> a - \int_{1-c/n}^1 \frac{1}{2}(1-a)\{n^2/(n-c)\}\{t/(1-c/n)\}^{n-1} dt \\ &= a - \frac{1}{2}(1-a)\{(1-c/n)^{-n} - 1\}, \end{aligned}$$

or $a < \{1 - (1-c/n)^n\} / \{1 + (1-c/n)^n\}$. This establishes the relation (2).

The above proof is valid for $0 < c < n$; however, for $c > 1$, the estimate just obtained is not as good as the estimate (3). In order to prove (3) we apply (4) with $f(z) = p_n(z)$, $r_1 = 1 - c/n$ where $1 < c \leq n$, $r_2 = 1 - 1/n$ and $\alpha = \alpha^*$ where $|p_n(z)|$ attains its minimum on the circle $\{z: |z| = 1 - 1/n\}$ at the point $z = (1 - 1/n)e^{i\alpha^*}$. We get

$$\min_{|z|=1-c/n} |p_n(z)| \leq |p_n((1-c/n)e^{i\alpha^*})| < \frac{(2n-1)c - (2n-c)(1-1/n)^n}{(2n-1)c + (2n-c)(1-1/n)^n}$$

which completes the proof of Theorem 1.

PROOF OF THEOREM 2. We consider the nonnegative trigonometric polynomial

$$\begin{aligned} t(\theta) &= (n+1)^{-2}[n(n+1) - 2\{n \cos \theta + (n-1)\cos 2\theta + \dots \\ &\quad + 2 \cos(n-1)\theta + \cos n\theta\}] \\ &\equiv 1 - \frac{1}{(n+1)^2} \left(\frac{\sin(n+1)\theta/2}{\sin \theta/2} \right)^2 \end{aligned}$$

of degree n vanishing at $\theta=0$. There exist (see [2, p. 117]) polynomials $p_n \in \mathcal{P}_{n,0}$ such that

$$(5) \quad |p_n(e^{i\theta})|^2 = t(\theta).$$

Amongst the various polynomials p_n satisfying (5) there is one (except for a constant factor of unit modulus) which does not vanish in U . If we

denote it by p_n^* , then for $r < 1$ and $-\pi \leq \varphi < \pi$

$$|p_n^*(r e^{i\varphi})| = \exp\left\{\frac{1}{4\pi} \int_{-\pi}^{\pi} \log |p_n^*(e^{i\theta})|^2 \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} d\theta\right\}.$$

Thus

$$|p_n^*((1 - c/n)e^{i\varphi})| = \exp(I_n(\varphi)),$$

where

$$I_n(\varphi) = \frac{1}{4\pi} (cn - \frac{1}{2}c^2) \int_{-\pi/2}^{\pi/2} \log |p_n^*(e^{2i\theta})|^2 \frac{d\theta}{\frac{1}{4}c^2 + (n^2 - cn)\sin^2(\theta - \frac{1}{2}\varphi)}.$$

It can be shown that for $0 \leq \theta \leq \pi/2$,

$$|p_n^*(e^{2i\theta})|^2 = 1 - \frac{1}{(n + 1)^2} \left(\frac{\sin(n + 1)\theta}{\sin \theta}\right)^2 = (1 + \gamma_n)D((n + 1)\theta)$$

where $D(u) = 1 - (\sin^2 u)/u^2$ and $|\gamma_n| < 5/(n + 1)^2$. Hence

$$(6) \quad I_n(\varphi) = -\frac{1}{4\pi} (cn - \frac{1}{2}c^2) \int_{-\pi/2}^{\pi/2} |\log D((n + 1)\theta)| \cdot \frac{d\theta}{\frac{1}{4}c^2 + (n^2 - cn)\sin^2(\theta - \frac{1}{2}\varphi)} + \delta_n,$$

where $|\delta_n| < 10/cn$ if $n \geq 3$. Since the right-hand side of (6) is decreased when $c^2/4 + (n^2 - cn)\sin^2(\theta - \varphi/2)$ is replaced by $c^2/4$ we conclude that

$$\begin{aligned} I_n(\varphi) &> -\frac{1}{\pi c} (n - \frac{1}{2}c) \int_{-\pi/2}^{\pi/2} |\log D((n + 1)\theta)| d\theta - |\delta_n| \\ &> -\frac{1}{\pi c} \frac{n - \frac{1}{2}c}{n + 1} \int_{-\infty}^{\infty} |\log D(u)| du - |\delta_n|, \end{aligned}$$

from which the statement of Theorem 2 follows.

With reference to the problem of Halász, it is natural to define a more general class $\mathcal{P}_{n,b}$ of polynomials $p_n(z)$ which are of degree at most n in z , satisfying $\max_{z \in T} |p_n(z)| = 1$, and $|p_n(1)| = b$ where $b \in [0, 1)$, and to estimate

$$\mu_b(c, n) = \sup_{p_n \in \mathcal{P}_{n,b}} \left\{ \min_{|z|=1-c/n} |p_n(z)| \right\}.$$

Our proof of Theorem 1 applies with slight modification, to give the following result.

THEOREM 1'. *If $p_n \in \mathcal{P}_{n,b}$, then for $0 < c < n$,*

$$\min_{|z|=1-c/n} |p_n(z)| < \frac{1 - (1 - 2b)(1 - c/n)^n}{1 + (1 - c/n)^n}.$$

Furthermore, if $c_0 \in (0, n)$ is arbitrary, and if $c_0 \leq c \leq n$, then

$$\min_{|z|=1-c/n} p(z) < \frac{A + \{2nb(c + c_0 - cc_0/n) - B\}(1 - c_0/n)^n}{A + \{2nb(c - c_0) + B\}(1 - c_0/n)^n}$$

where $A = (2n - c_0)c$, $B = (2n - c)c_0$.

In analogy with the problem of Halász, or the more general case just considered, let $\mathcal{F}_{n,b}$ denote the class of all polynomials $p_n(z)$ of degree at most n in z which satisfy $\max_{z \in T} \operatorname{Re} p_n(z) = 1$, and $\operatorname{Re} p_n(1) = b$, where $b \in [0, 1)$.

THEOREM 1". If $p_n \in \mathcal{F}_{n,b}$, then

$$(7) \quad \min_{|z|=1-c/n} \operatorname{Re} p_n(z) < B(c) \equiv \frac{1 - (1 - 2b)(1 - c/n)^n}{1 + (1 - c/n)^n}.$$

Furthermore, for any fixed $c_1 \in (0, n)$ and for $c_1 \leq c \leq n$,

$$(8) \quad \min_{|z|=1-c/n} \operatorname{Re} p_n(z) < 1 + \log \left\{ \left(1 - \frac{(2n - c)c_1 e - e^A}{(2n - c_1)c e + e^A} \right) / \left(1 + \frac{(2n - c)c_1 e - e^A}{(2n - c_1)c e + e^A} \right) \right\}$$

where $A = B(c_1)$.

SKETCH OF PROOF. The inequality (7) can be proved in the same way as (2). If $\operatorname{Re} p_n(z)$ attains its minimum on the circle $\{z: |z|=1-c_1/n\}$ at $z = (1 - c_1/n)e^{i\alpha_1}$, then for $c_1 < c \leq n$, we may apply Lemma 2 with $f(z) = \exp\{p_n(z)\} - 1$, $r_1 = 1 - c/n$, $r_2 = 1 - c_1/n$ and $\alpha = \alpha_1$, to get

$$\exp \operatorname{Re}\{p_n((1 - c/n)e^{i\alpha_1}) - 1\} \leq (B - C)/(B + C)$$

where

$$B = (2n - c_1)c[1 + \exp\{\operatorname{Re}((1 - c_1/n)e^{i\alpha_1}) - 1\}],$$

$$C = (2n - c)c_1[1 - \exp\{p_n((1 - c_1/n)e^{i\alpha_1}) - 1\}].$$

The inequality (8) now follows from this, in view of the definition of A , and since

$$\min_{|z|=1-c/n} \operatorname{Re} p_n(z) \leq \operatorname{Re} p_n((1 - c/n)e^{i\alpha_1}).$$

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