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# AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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ABSTRACT. Let $\mathscr{P}_{n,b}$ denote the class of all polynomials $p_n(z)$ of degree at most $n$ in $z$ which satisfy $\max_{|z|=1}|p_n(z)|=1$, and $|p_n(1)|=b$, $0\leqq b<1$. Let $c\in(0,n]$, and set

$$\mu_b(c,n)=\sup_{p_n\in\mathscr{P}_{n,b}}\left\{\min_{|z|=1-c/n}|p_n(z)|\right\}.$$

Upper estimates for $\mu_b(c,n)$ are obtained.

Let $U$ denote the open unit disc in the complex $z$ plane, $T$ its boundary, and let $\mathscr{P}_{n,0}$ denote the class of all polynomials $p_n(z)$ of degree at most $n$ in $z$, satisfying $\max_{z\in T}|p_n(z)|=1$ and $p_n(1)=0$. The extremal problem in question is to estimate

$$\mu(c,n)=\sup_{p_n\in\mathscr{P}_{n,0}}\left\{\min_{|z|=1-c/n}|p_n(z)|\right\},$$

where $0<c\leqq n$. This problem was mentioned by Professor Paul Erdös during a lecture at the University of Montreal in July, 1971. He attributed the problem to G. Halász, of the Mathematical Institute of the Hungarian Academy of Sciences; Erdös asked if there exists a constant $c$ such that $\mu(c,n)=1-\varepsilon_n$ where $\varepsilon_n\to 0$ as $n\to\infty$.

It is easily seen that no such constant $c$ exists. In fact, if $p_n\in\mathscr{P}_{n,0}$, then also $q_n\in\mathscr{P}_{n,0}$, where $q_n(z)=z^n p_n(1/z)$, and by S. Bernstein's theorem [3, p. 45] on the derivative of a polynomial, $|q_n'(z)|\leqq n$ for $z\in T$. Hence it follows that $|z^{n-1}q_n'(1/z)|\leqq n$ for $z\in T$ and by the maximum principle, also for all $z\in U$. Replacing $z$ by $1/z$ we find that

$$|q_n'(z)|\leqq n|z|^{n-1}\quad\text{for all } |z|\geqq 1.$$

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Consequently,

$$|q_n((1-c/n)^{-1})| = \left|\int_1^{(1-c/n)^{-1}} q_n'(t)\,dt\right|$$
$$\leqq (1-c/n)^{-n} - 1 = (1-c/n)^{-n}\{1-(1-c/n)^n\},$$

that is,

$$(1)\qquad \begin{aligned}|p_n(1-c/n)| &= |(1-c/n)^n q_n((1-c/n)^{-1})|\\ &\leqq 1-(1-c/n)^n \to 1-e^{-c} \quad \text{as } n\to\infty.\end{aligned}$$

The inequality (1) provides a negative answer to the question raised by Erdös and also gives an upper estimate for $\mu(c, n)$. However, this estimate is quite crude. The following theorem, which we shall prove, gives "essentially" best possible upper estimates for $\mu(c, n)$.

THEOREM 1. *In the above notation,*

$$(2)\qquad \mu(c,n) < \{1-(1-c/n)^n\}/\{1+(1-c/n)^n\} \quad \textit{if } 0 < c \leqq 1,$$

*and*

$$(3)\qquad \mu(c,n) < \frac{\{(2n-1)c-(2n-c)(1-1/n)^n\}}{\{(2n-1)c+(2n-c)(1-1/n)^n\}} \quad \textit{if } 1 < c \leqq n.$$

The right-hand side of (2) is equal to $c/2+o(c)$ as $c\to 0$; moreover, the polynomial $p_n(z)=(1-z^n)/2$ satisfies

$$\min_{|z|=1-c/n} |p_n(z)| = |p_n(1-c/n)| = \{1-(1-c/n)^n\}/2 = c/2+o(c)$$

as $c\to 0$. Consequently, the inequality (2) is the best possible in the limit as $c\to 0$.

We find from (3) that $\mu(c,n)\leqq 1-1/ec+o(1/c)$ as $c\to\infty$. We shall show that the function $1/(ec)$ cannot be replaced by one which approaches zero more slowly with regards to order, as $c\to\infty$. We prove

THEOREM 2. *Given*

$$\lambda > \frac{1}{\pi}\int_{-\infty}^{\infty} \left|\log\left(1-\frac{\sin^2 u}{u^2}\right)\right| du$$

*there exists a positive number $A(\lambda)$, depending only on $\lambda$, such that whenever $c>A(\lambda)$, then*

$$\mu(c,n) > \exp(-\lambda/c) > 1-\lambda/c.$$

For the proof of Theorem 1 we use two subsidiary results.

LEMMA 1 [1, THEOREM 4]. *Let $D$ be a circular domain in the z-plane, and $S$ an arbitrary set of points in the w-plane. If the polynomial $p_n$ of degree* *n satisfies* $p_n(z)=w \in S$ *for all* $z \in D$, *then for all* $z \in D$ *and all* $\zeta \in D$,

$$\frac{\zeta p_n'(z)}{n} + p_n(z) - \frac{zp_n'(z)}{n} \in S.$$

LEMMA 2. *If* $f(z)$ *is analytic in* $U$, *where it satisfies* $|f(z)| \leqq 1$, *then for* $0 \leqq \alpha < 2\pi$ *and* $0 \leqq r_1 < r_2 < 1$,

$$f(r_1e^{i\alpha}) \leqq (A - B)/(A + B) \tag{4}$$

*where*

$$A = (1 + r_2)(1 - r_1)\{1 + |f(r_2e^{i\alpha})|\},$$
$$B = (1 - r_2)(1 + r_1)\{1 - |f(r_2e^{i\alpha})|\}.$$

PROOF OF LEMMA 2. It is well known that if $f(z)$ is analytic in $U$, where it satisfies $|f(z)| \leqq 1$, then

$$|f'(z)|/(1 - |f(z)|^2) \leqq 1/(1 - |z|^2) \quad \text{for all } z \in U.$$

Hence

$$\left|\int_{r_1}^{r_2} \frac{(d/dr)\,|f(re^{i\alpha})|}{1 - |f(re^{i\alpha})|^2}\,dr\right| \leqq \int_{r_1}^{r_2} \frac{|f'(re^{i\alpha})|}{1 - |f(re^{i\alpha})|^2}\,dr \leqq \int_{r_1}^{r_2} \frac{dr}{1 - r^2}.$$

Now if $|f(r_1e^{i\alpha})| > |f(r_2e^{i\alpha})|$, we get

$$\left\{\frac{1 + G_1}{1 - G_1}\right\} \Big/ \left\{\frac{1 + G_2}{1 - G_2}\right\} \leqq \left\{\frac{1 + r_2}{1 - r_2}\right\} \Big/ \left\{\frac{1 + r_1}{1 - r_1}\right\}$$

where $G_k = |f(r_ke^{i\alpha})|$, $k = 1, 2$, which readily gives the desired estimate of $|f(r_1e^{i\alpha})|$. The inequality (4) is trivially true if $|f(r_1e^{i\alpha})| \leqq |f(r_2e^{i\alpha})|$.

PROOF OF THEOREM 1. Let $p_n \in \mathscr{P}_{n,0}$, $0 < c \leqq 1$, and let

$$\min_{|z|=1-c/n} |p_n(z)| = a.$$

We wish to show that

$$a < \{1 - (1 - c/n)^n\}/\{1 + (1 - c/n)^n\}.$$

Without loss of generality we may suppose that $p_n(z) \neq 0$ in $U$, and therefore

$$\min_{|z| \leqq 1-c/n} |p_n(z)| = \min_{|z|=1-c/n} |p_n(z)| = a.$$

This implies that $p_n$ maps the circular domain $D = \{z : |z| \leqq 1 - c/n\}$ onto a set $S$ which lies in the ring $\{w : a \leqq |w| < 1\}$. Hence by Lemma 1,

$$(1 - c/n)\,|p_n'(z)|/n < (1 - a)/2$$

for all $|z|=1-c/n$, i.e.,

$$|p_n'((1-c/n)z)| < \tfrac{1}{2}(1-a)n^2/(n-c) \quad \text{for all } |z|=1.$$

The same inequality holds for the polynomial $z^{n-1}p_n'((1-c/n)/z)$. Using the maximum modulus principle, we therefore conclude that

$$|z^{n-1}p_n'((1-c/n)/z)| \leqq \tfrac{1}{2}(1-a)n^2/(n-c) \quad \text{for all } z\in(U\cup T).$$

Replacing $z$ by $(1-c/n)/z$ we obtain

$$|p_n'(z)| < \tfrac{1}{2}\{(1-a)n^2/(n-c)\}\{z/(1-c/n)\}^{n-1} \quad \text{for all } |z|\geqq 1-c/n.$$

This implies that

$$\begin{aligned} 0 = |p_n(1)| &= \left|p_n(1-c/n) + \int_{1-c/n}^{1} p_n'(t)\,dt\right| \\ &> a - \int_{1-c/n}^{1} \tfrac{1}{2}(1-a)\{n^2/(n-c)\}\{t/(1-c/n)\}^{n-1}\,dt \\ &= a - \tfrac{1}{2}(1-a)\{(1-c/n)^{-n}-1\}, \end{aligned}$$

or $a<\{1-(1-c/n)^n\}/\{1+(1-c/n)^n\}$. This establishes the relation (2).

The above proof is valid for $0<c<n$; *however, for* $c>1$, *the estimate just obtained is not as good as the estimate* (3). In order to prove (3) we apply (4) with $f(z)=p_n(z)$, $r_1=1-c/n$ where $1<c\leqq n$, $r_2=1-1/n$ and $\alpha=\alpha^*$ where $|p_n(z)|$ attains its minimum on the circle $\{z:|z|=1-1/n\}$ at the point $z=(1-1/n)e^{i\alpha^*}$. We get

$$\min_{|z|=1-c/n} |p_n(z)| \leqq |p_n((1-c/n)e^{i\alpha^*})| < \frac{(2n-1)c-(2n-c)(1-1/n)^n}{(2n-1)c+(2n-c)(1-1/n)^n}$$

which completes the proof of Theorem 1.

Proof of Theorem 2. We consider the nonnegative trigonometric polynomial

$$\begin{aligned} t(\theta) &= (n+1)^{-2}[n(n+1) - 2\{n\cos\theta + (n-1)\cos 2\theta + \cdots \\ &\qquad + 2\cos(n-1)\theta + \cos n\theta\}] \\ &\equiv 1 - \frac{1}{(n+1)^2}\left(\frac{\sin(n+1)\theta/2}{\sin\theta/2}\right)^2 \end{aligned}$$

of degree $n$ vanishing at $\theta=0$. There exist (see [2, p. 117]) polynomials $p_n\in\mathscr{P}_{n,0}$ such that

$$(5) \qquad |p_n(e^{i\theta})|^2 = t(\theta).$$

Amongst the various polynomials $p_n$ satisfying (5) there is one (except for a constant factor of unit modulus) which does not vanish in $U$. If we denote it by $p_n^*$, then for $r<1$ and $-\pi\leqq\varphi<\pi$

$$|p_n^*(re^{i\varphi})| = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log|p_n^*(e^{i\theta})|^2\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\theta\right\}.$$

Thus

$$|p_n^*((1-c/n)e^{i\varphi})| = \exp(I_n(\varphi)),$$

where

$$I_n(\varphi) = \frac{1}{4\pi}(cn-\tfrac{1}{2}c^2)\int_{-\pi/2}^{\pi/2}\log|p_n^*(e^{2i\theta})|^2\frac{d\theta}{\frac{1}{4}c^2+(n^2-cn)\sin^2(\theta-\frac{1}{2}\varphi)}.$$

It can be shown that for $0\leqq\theta\leqq\pi/2$,

$$|p_n^*(e^{2i\theta})|^2 = 1-\frac{1}{(n+1)^2}\left(\frac{\sin(n+1)\theta}{\sin\theta}\right)^2 = (1+\gamma_n)D((n+1)\theta)$$

where $D(u)=1-(\sin^2 u)/u^2$ and $|\gamma_n|<5/(n+1)^2$. Hence

$$I_n(\varphi) = -\frac{1}{4\pi}(cn-\tfrac{1}{2}c^2)\int_{-\pi/2}^{\pi/2}|\log D((n+1)\theta)| \cdot\frac{d\theta}{\frac{1}{4}c^2+(n^2-cn)\sin^2(\theta-\frac{1}{2}\varphi)}+\delta_n, \tag{6}$$

where $|\delta_n|<10/cn$ if $n\geqq3$. Since the right-hand side of (6) is decreased when $c^2/4+(n^2-cn)\sin^2(\theta-\varphi/2)$ is replaced by $c^2/4$ we conclude that

$$\begin{aligned} I_n(\varphi) &> -\frac{1}{\pi c}(n-\tfrac{1}{2}c)\int_{-\pi/2}^{\pi/2}|\log D((n+1)\theta)|\,d\theta-|\delta_n| \\ &> -\frac{1}{\pi c}\frac{n-\frac{1}{2}c}{n+1}\int_{-\infty}^{\infty}|\log D(u)|\,du-|\delta_n|, \end{aligned}$$

from which the statement of Theorem 2 follows.

With reference to the problem of Halász, it is natural to define a more general class $\mathscr{P}_{n,b}$ of polynomials $p_n(z)$ which are of degree at most $n$ in $z$, satisfying $\max_{z\in T}|p_n(z)|=1$, and $|p_n(1)|=b$ where $b\in[0,1)$, and to estimate

$$\mu_b(c,n) = \sup_{p_n\in\mathscr{P}_{n,b}}\left\{\min_{|z|=1-c/n}|p_n(z)|\right\}.$$

Our proof of Theorem 1 applies with slight modification, to give the following result.

THEOREM 1′. *If $p_n\in\mathscr{P}_{n,b}$, then for $0<c<n$,*

$$\min_{|z|=1-c/n}|p_n(z)| < \frac{1-(1-2b)(1-c/n)^n}{1+(1-c/n)^n}.$$

*Furthermore, if $c_0 \in (0, n)$ is arbitrary, and if $c_0 \leqq c \leqq n$, then*

$$\min_{|z|=1-c/n} p(z) < \frac{A + \{2nb(c + c_0 - cc_0/n) - B\}(1 - c_0/n)^n}{A + \{2nb(c - c_0) + B\}(1 - c_0/n)^n}$$

*where $A=(2n-c_0)c$, $B=(2n-c)c_0$.*

In analogy with the problem of Halász, or the more general case just considered, let $\mathscr{F}_{n,b}$ denote the class of all polynomials $p_n(z)$ of degree at most $n$ in $z$ which satisfy $\max_{z\in T} \operatorname{Re} p_n(z)=1$, and $\operatorname{Re} p_n(1)=b$, where $b \in [0, 1)$.

THEOREM 1″. *If $p_n \in \mathscr{F}_{n,b}$, then*

$$\min_{|z|=1-c/n} \operatorname{Re} p_n(z) < B(c) \equiv \frac{1 - (1 - 2b)(1 - c/n)^n}{1 + (1 - c/n)^n}. \tag{7}$$

*Furthermore, for any fixed $c_1 \in (0, n)$ and for $c_1 \leqq c \leqq n$,*

$$\begin{aligned} &\min_{|z|=1-c/n} \operatorname{Re} p_n(z) \\ &\quad < 1 + \log\left\{\left(1 - \frac{(2n - c)c_1}{(2n - c_1)c}\frac{e - e^A}{e + e^A}\right)\Big/\left(1 + \frac{(2n - c)c_1}{(2n - c_1)c}\frac{e - e^A}{e + e^A}\right)\right\} \end{aligned} \tag{8}$$

*where $A=B(c_1)$.*

SKETCH OF PROOF. The inequality (7) can be proved in the same way as (2). If $\operatorname{Re} p_n(z)$ attains its minimum on the circle $\{z:|z|=1-c_1/n\}$ at $z=(1-c_1/n)e^{i\alpha_1}$, then for $c_1<c\leqq n$, we may apply Lemma 2 with $f(z)=\exp\{p_n(z)\}-1$, $r_1=1-c/n$, $r_2=1-c_1/n$ and $\alpha=\alpha_1$, to get

$$\exp \operatorname{Re}\{p_n((1 - c/n)e^{i\alpha_1}) - 1\} \leqq (B - C)/(B + C)$$

where

$$B = (2n - c_1)c[1 + \exp\{\operatorname{Re}((1 - c_1/n)e^{i\alpha_1}) - 1\}],$$

$$C = (2n - c)c_1[1 - \exp\{p_n((1 - c_1/n)e^{i\alpha_1}) - 1\}].$$

The inequality (8) now follows from this, in view of the definition of $A$, and since

$$\min_{|z|=1-c/n} \operatorname{Re} p_n(z) \leqq \operatorname{Re} p_n((1 - c/n)e^{i\alpha_1}).$$

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