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# OSCILLATORY BEHAVIOR OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. It is shown that if $p(x) \leqq 0$, $q(x) > 0$ and if $y''' + py' + qy = 0$ has an oscillatory solution then every nonoscillatory solution is a constant multiple of one nonoscillatory solution.

A solution of

$$y''' + p(x)y' + q(x)y = 0 \tag{1}$$

will be said to be oscillatory if it changes signs for arbitrarily large values of $x$. Other solutions will be said to be nonoscillatory. It will be assumed that $p(x)$, $q(x)$, and $p'(x)$ are continuous on $[0, +\infty)$.

The first theorem will be in the setting of Class I or Class II equations as defined by Hanan **[3]**.

THEOREM 1. *Suppose* (1) *is of Class* I *or Class* II. *If* (1) *has an oscillatory solution and if $N$ is a nontrivial nonoscillatory solution of its adjoint*

$$y''' + p(x)y' + (p'(x) - q(x))y = 0 \tag{2}$$

*then there are two independent oscillatory solutions of* (1) *that satisfy*

$$\left(\frac{y'}{N}\right)' + \left(\frac{N'' + pN}{N^2}\right)y = 0. \tag{3}$$

PROOF. Since (1) is of Class I or Class II, so is (2) **[3]**. Thus, if $N$ is a nontrivial nonoscillatory solution of (2) there is an $a \in [0, +\infty)$ such that $N(x) \neq 0$ for $x > a$. Further, since (1) has an oscillatory solution, there are two independent oscillatory solutions $y_1$ and $y_2$ of (2) **[5]**. It is easily verified that $y_1N' - Ny_1'$ and $y_2N' - Ny_2'$ are independent oscillatory solutions of (1) and (3).

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COROLLARY. *Suppose* (1) *is of Class* I *or* II *and has an oscillatory solution. If N and y are independent solutions of* (2) *such that N is nonoscillatory, then* $Ny'-yN'$ *is an oscillatory solution of* (1).

PROOF. Under the conditions of the Corollary (3) is oscillatory and $Ny'-yN'$ is a solution of (3).

The proof of the following theorem is essentially contained in the proof Theorem 1.5 [**6**], but is included here for completeness.

THEOREM 2. *Suppose* $p(x)\leqq 0$, $q(x)>0$ *and that* (1) *has an oscillatory solution. Suppose* $N(x)$ *is a solution of* (2) *defined by* $N(a)=N'(a)=0$, $N''(a)=1$ *for* $a\in[0,+\infty)$. *Then* $N(x)>0$, $N'(x)>0$ *and* $N''(x)>N''(x)+p(x)N(x)>1$ *for* $x>a$.

PROOF. By [**6**], (1) is Class I. Thus (2) is Class II [**3**]. It follows that $N(x)>0$ for $x>a$. Now $(N''(x)+p(x)N(x))'=q(x)N(x)>0$ for $x>a$. Thus since $N''+pN$ is an increasing function of $x$ for $x>a$ and since $p(x)\leqq 0$, $N''(x)>N''(x)+p(x)N(x)>N''(a)+p(a)N(a)=1$ for $x>a$. It now follows that $N'(x)>0$ for all $x>a$.

THEOREM 3. *Suppose* (1) *is Class* I *or* II, *that* $q(x)>0$ *and that* (2) *has a nonoscillatory solution N such that* $N(x)>0$ *and* $N'(x)>0$ *for* $x>a$. *Then*

$$G[y(x)]\equiv Ny'^2+(N''+pN)y^2$$

*is an increasing function of x for* $x>a$, *where y is any solution of* (3).

PROOF.

$$\begin{aligned} G'[y(x)] &= 2Ny'y''+N'y'^2+2y(N''+pN)y'+qNy^2 \\ &= 2y'[N'y'-(N''+pN)y]+N'y'^2+2yy'(N''+pN)+qNy^2 \\ &= 3N'y'^2+qNy^2>0 \quad \text{for } x>a. \end{aligned}$$

Thus, the result follows.

Our main result which generalizes results of Lazer [**6**] and Gregus [**2**] now follows.

THEOREM 4. *If* $p(x)\leqq 0$, $q(x)>0$ *and* (1) *has an oscillatory solution then every nonoscillatory solution is a constant multiple of one nonoscillatory solution.*

PROOF. Let $N$ be a solution of (2) defined by $N(a)=N'(a)=0$, $N''(a)=1$ for $a\in[0,+\infty)$. Since $p(x)\leqq 0$ and $q(x)>0$, (1) is Class I and has a solution $z(x)$ such that $z(x)>0$, $z'(x)<0$, $z''(x)>0$ for all $x\in[0,+\infty)$ [**6**]. Let $u_1$ and $u_2$ be independent solutions of (1) that satisfy (3). Then $z$, $u_1$, and $u_2$ is a basis for the solution space of (1). Assuming that there are two independent solutions of (1) that are nonoscillatory then $z+c_1u_1+c_2u_2$ is a nonoscillatory solution of (1) for some $c_1$ and $c_2$ not both zero. Let $-y_1=c_1u_1+c_2u_2$ and let $y_2$ be from the space spanned by $\{u_1, u_2\}$ independent from $y_1$. By [**6**], $|z(x)-y_1(x)|>0$. Since $y_1$ is oscillatory and $z(x)>0$ it is clear that $z(x)-y_1(x)>0$ for $x \in [0, +\infty)$.

Since $y_1$, $y_2$, $z$ are independent solutions of (1)

$$0 \neq k = \begin{vmatrix} y_1 & y_2 & z \\ y_1' & y_2' & z' \\ y_1'' & y_2'' & z'' \end{vmatrix}$$

where $k$ is a constant.

Expanding, we obtain

$$z[N'' + pN] - z'N' + z''N = k_1 \neq 0$$

where $k_1$ is a constant. By the observation about $z$ noted above and Theorem 2, $z[N''+pN]$, $-z'N'$ and $z''N$ are each positive for $x>a$. Thus $0<z[N''+pN]<k_1$ for $x>a$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $y_1'(x_n)=0$ and $y_1''(x_n)<0$ such that $x_n\to\infty$. Then

$$\begin{aligned} k_1 &> z(x_n)[N''(x_n) + p(x_n)N(x_n)] \\ &\geqq y_1(x_n)[N''(x_n) + p(x_n)N(x_n)] > 0. \end{aligned}$$

But, by [**4**], $\lim_{x\to\infty} z(x)=0$. Therefore $y_1^2(x_n)[N''(x_n)+p(x_n)N(x_n)]\to 0$ as $n\to\infty$. But this is not possible since $G[y_1(x)]$ in Theorem 3 is increasing.

The following result gives a condition for certain equations of Class II to have behavior similar to that observed by Ahmad and Lazer in [**1**].

THEOREM 5. *If* $p(x)\leqq 0$, $q(x)-p'(x)<0$ *and* (1) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions of* (1) *whose zeros separate and such that a solution of* (1) *is oscillatory if and only if it is a nontrivial linear combination of them.*

PROOF. Since $p(x)\leqq 0$ and $p'(x)-q(x)>0$, there is a solution $N$ of (2) such that $N(x)>0$ for all $x \in [0, +\infty)$ [**6**]. Thus by Theorem 1 there are two linearly independent oscillatory solutions, $y_1$ and $y_2$, of (1) whose zeros separate.

Suppose there is an oscillatory solution of (1) that is not a linear combination of $y_1$ and $y_2$. Then by [**5**] there are two independent nonoscillatory solutions of (2), but this is contrary to Theorem 4.

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