OSCILLATORY BEHAVIOR OF THIRD ORDER
DIFFERENTIAL EQUATIONS

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Abstract. It is shown that if \( p(x) \leq 0 \), \( q(x) > 0 \) and if \( y'' + py' + qy = 0 \) has an oscillatory solution then every nonoscillatory solution is a constant multiple of one nonoscillatory solution.

A solution of

\[
y'' + p(x)y' + q(x)y = 0
\]

will be said to be oscillatory if it changes signs for arbitrarily large values of \( x \). Other solutions will be said to be nonoscillatory. It will be assumed that \( p(x), q(x), \) and \( p'(x) \) are continuous on \([0, +\infty)\).

The first theorem will be in the setting of Class I or Class II equations as defined by Hanan [3].

Theorem 1. Suppose (1) is of Class I or Class II. If (1) has an oscillatory solution and if \( N \) is a nontrivial nonoscillatory solution of its adjoint

\[
y'' + p(x)y' + (p'(x) - q(x))y = 0
\]

then there are two independent oscillatory solutions of (1) that satisfy

\[
\left( \frac{y'}{N} \right)' + \left( \frac{N'' + pN}{N^2} \right)y = 0.
\]

Proof. Since (1) is of Class I or Class II, so is (2) [3]. Thus, if \( N \) is a nontrivial nonoscillatory solution of (2) there is an \( a \in [0, +\infty) \) such that \( N(x) \neq 0 \) for \( x > a \). Further, since (1) has an oscillatory solution, there are two independent oscillatory solutions \( y_1 \) and \( y_2 \) of (2) [5]. It is easily verified that \( y_1 N' - Ny_1' \) and \( y_2 N' - Ny_2' \) are independent oscillatory solutions of (1) and (3).


Key words and phrases. Differential equations, third order, oscillation, basis of solutions.
COROLLARY. Suppose (1) is of Class I or II and has an oscillatory solution. If \( N \) and \( y \) are independent solutions of (2) such that \( N \) is non-oscillatory, then \( Ny' - yN' \) is an oscillatory solution of (1).

PROOF. Under the conditions of the Corollary (3) is oscillatory and \( Ny' - yN' \) is a solution of (3).

The proof of the following theorem is essentially contained in the proof Theorem 1.5 [6], but is included here for completeness.

THEOREM 2. Suppose \( p(x) \leq 0, q(x) > 0 \) and that (1) has an oscillatory solution. Suppose \( N(x) \) is a solution of (2) defined by \( N(a) = N'(a) = 0, N''(a) = 1 \) for \( a \in [0, +\infty) \). Then \( N(x) > 0, N'(x) > 0 \) and \( N''(x) > N''(x) + p(x)N(x) > 1 \) for \( x > a \).

PROOF. By [6], (1) is Class I. Thus (2) is Class II [3]. It follows that \( N(x) > 0 \) for \( x > a \). Now \( (N''(x) + p(x)N(x))' = q(x)N(x) > 0 \) for \( x > a \). Thus since \( N'' + pN \) is an increasing function of \( x \) for \( x > a \) and since \( p(x) \leq 0 \), \( N''(x) > N''(x) + p(x)N(x) > N''(a) + p(a)N(a) = 1 \) for \( x > a \). It now follows that \( N'(x) > 0 \) for all \( x > a \).

THEOREM 3. Suppose (1) is Class I or II, that \( q(x) > 0 \) and that (2) has a nonoscillatory solution \( N \) such that \( N(x) > 0 \) and \( N'(x) > 0 \) for \( x > a \). Then

\[
G[y(x)] = Ny'^2 + (N'' + pN)y^2
\]

is an increasing function of \( x \) for \( x > a \), where \( y \) is any solution of (3).

PROOF.

\[
G'[y(x)] = 2Ny'y'' + N'y'^2 + 2y(N'' + pN)y' + qNy^2
\]

\[
= 2y'[N'y' - (N'' + pN)y] + N'y'^2 + 2yy'(N'' + pN) + qNy^2
\]

\[
= 3N'y'^2 + qNy^2 > 0 \text{ for } x > a.
\]

Thus, the result follows.


THEOREM 4. If \( p(x) \leq 0, q(x) > 0 \) and (1) has an oscillatory solution then every nonoscillatory solution is a constant multiple of one nonoscillatory solution.

PROOF. Let \( N \) be a solution of (2) defined by \( N(a) = N'(a) = 0, N''(a) = 1 \) for \( a \in [0, +\infty) \). Since \( p(x) \leq 0 \) and \( q(x) > 0 \), (1) is Class I and has a solution \( z(x) \) such that \( z(x) > 0, z'(x) < 0, z''(x) > 0 \) for all \( x \in [0, +\infty) \) [6]. Let \( u_1 \) and \( u_2 \) be independent solutions of (1) that satisfy (3). Then \( z, u_1, \) and \( u_2 \) is a basis for the solution space of (1). Assuming that there
are two independent solutions of (1) that are nonoscillatory then \( z + c_1u_1 + c_2u_2 \) is a nonoscillatory solution of (1) for some \( c_1 \) and \( c_2 \) not both zero. Let \(-y_1 = c_1u_1 + c_2u_2\) and let \( y_2 \) be from the space spanned by \( \{u_1, u_2\} \) independent from \( y_1 \). By [6], \(|z(x) - y_1(x)| > 0\). Since \( y_1 \) is oscillatory and \( z(x) > 0 \) it is clear that \( z(x) - y_1(x) > 0 \) for \( x \in [0, +\infty) \).

Since \( y_1, y_2, z \) are independent solutions of (1)

\[
0 \neq k = \begin{vmatrix}
y_1 & y_2 & z \\
y_1' & y_2' & z' \\
y_1'' & y_2'' & z''
\end{vmatrix}
\]

where \( k \) is a constant.

Expanding, we obtain

\[
z[N'' + pN] - z'N' + z''N = k_1 \neq 0
\]

where \( k_1 \) is a constant. By the observation about \( z \) noted above and Theorem 2, \( z[N'' + pN], -z'N' \) and \( z''N \) are each positive for \( x > a \). Thus \( 0 < z[N'' + pN] < k_1 \) for \( x > a \). Let \( \{x_n\}_{n=1}^\infty \) be a sequence such that \( y_1'(x_n) = 0 \) and \( y_1''(x_n) < 0 \) such that \( x_n \to \infty \). Then

\[
k_1 > z(x_n)[N''(x_n) + p(x_n)N(x_n)]
\]

\[
\geq y_1(x_n)[N''(x_n) + p(x_n)N(x_n)] > 0.
\]

But, by [4], \( \lim_{x \to \infty} z(x) = 0 \). Therefore \( y_1'(x_n)[N''(x_n) + p(x_n)N(x_n)] \to 0 \) as \( n \to \infty \). But this is not possible since \( G[y_1(x)] \) in Theorem 3 is increasing.

The following result gives a condition for certain equations of Class II to have behavior similar to that observed by Ahmad and Lazer in [1].

**Theorem 5.** If \( p(x) \leq 0, q(x) - p'(x) < 0 \) and (1) has an oscillatory solution, then there exist two linearly independent oscillatory solutions of (1) whose zeros separate and such that a solution of (1) is oscillatory if and only if it is a nontrivial linear combination of them.

**Proof.** Since \( p(x) \leq 0 \) and \( p'(x) - q(x) > 0 \), there is a solution \( N \) of (2) such that \( N(x) > 0 \) for all \( x \in [0, +\infty) \) [6]. Thus by Theorem 1 there are two linearly independent oscillatory solutions, \( y_1 \) and \( y_2 \), of (1) whose zeros separate.

Suppose there is an oscillatory solution of (1) that is not a linear combination of \( y_1 \) and \( y_2 \). Then by [5] there are two independent nonoscillatory solutions of (2), but this is contrary to Theorem 4.

**References**

2. M. Gregus, *Über einige Eigenschaften der Lösungen der Differentialgleichungen*
   \[ y'' + 2Ay' + (A' + b)y = 0, \ A \leq 0, \] 


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