

## LEFT CENTRALIZERS OF AN $H^*$ -ALGEBRA

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**ABSTRACT.** An explicit characterization is given of the left centralizers of a proper  $H^*$ -algebra  $A$ . Each left centralizer is seen to correspond to a bounded family of bounded operators, where each operator acts on a Hilbert space associated with a minimal-closed two-sided ideal of  $A$ .

**Introduction.** Let  $A$  be a semisimple Banach algebra. As in [3], we call a linear operator  $T$  on  $A$  a *left centralizer* if

$$T(xy) = T(x)y, \quad x, y \in A.$$

In this note we give an explicit characterization of the left centralizers on  $A$  when  $A$  is a proper  $H^*$ -algebra. Centralizers on  $H^*$ -algebras have been considered in [1], [6], and [9]. The same characterization holds when  $A$  is a dual  $B^*$ -algebra, and has been given by Malviya and Tomiuk in [7]. Our proof is similar to that in [7]. We include most details for the sake of completeness.

Use will be made of the structure theory of  $H^*$ -algebras (see e.g. [8]), which we shall review here briefly, after introducing some notation.

Given a family of Banach algebras,  $\{A_\gamma\}_{\gamma \in \Gamma}$ , and numbers  $k_\gamma \geq 1$ , we denote by  $l^p(\{A_\gamma, k_\gamma\})$ ,  $1 \leq p < \infty$ , the set of functions  $x$  on  $\Gamma$  with  $x(\gamma) \in A_\gamma$  and

$$\|x\|_p = \left( \sum_\gamma k_\gamma^p \|x(\gamma)\|^p \right)^{1/p} < \infty.$$

We denote by  $l^\infty(\{A_\gamma\})$  the set of functions  $x$  on  $\Gamma$  with  $x(\gamma) \in A_\gamma$  and

$$\|x\|_\infty = \sup_\gamma \|x(\gamma)\| < \infty.$$

With the usual operations for functions and the norm  $\|x\|_p$ , the above sets become Banach algebras. We denote by  $c_0(\{A_\gamma\})$  the closed subalgebra of  $l^\infty(\{A_\gamma\})$  consisting of those functions  $x$  for which  $\{\gamma: \|x(\gamma)\| \geq \varepsilon\}$  is finite for all  $\varepsilon > 0$ .

Given a Hilbert space  $H$ ,  $B(H)$  denotes the algebra of bounded linear operators on  $H$ , endowed with the operator norm,  $\|\cdot\|_0$ ;  $B_c(H)$  denotes

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the closed two-sided ideal of compact operators. We denote by  $B_s(H)$  the two-sided ideal of Hilbert-Schmidt operators, endowed with the Hilbert-Schmidt norm,  $\| \cdot \|_s$ . With this latter norm,  $B_s(H)$  is a Banach algebra (see [8] or [11]).

If  $A$  is a proper  $H^*$ -algebra, let  $\{A_\gamma\}_{\gamma \in \Gamma}$  denote its collection of minimal-closed two-sided ideals. For each  $\gamma$ , let  $H_\gamma$  be some minimal left ideal of  $A_\gamma$ . Then, under the left regular representation,  $A_\gamma$  is isomorphic to  $B_s(H_\gamma)$ , and there exist  $k_\gamma \geq 1$  such that  $A$  is isometrically isomorphic to  $l^2(\{B_s(H_\gamma), k_\gamma\})$ . We denote this isomorphism by  $a \rightarrow \hat{a}$ . For  $S \subset A$ , let  $\hat{S} = \{\hat{a} : a \in S\}$ . Then  $\hat{A}_\gamma = \{x \in l^2(\{B_s(H_\beta), k_\beta\}) : x(\beta) = 0, \beta \neq \gamma\}$ .

**The main result.** If  $A$  is a semisimple Banach algebra, we denote the left centralizers on  $A$  by  $\mathcal{L}(A)$ . A theorem of Johnson and Sinclair states that any left centralizer on  $A$  is continuous [4]. When endowed with the operator norm,  $\mathcal{L}(A)$  is a Banach algebra. We denote this norm simply by  $\| \cdot \|$ . When  $A$  is a left ideal in a Banach algebra  $B$ , then, for  $y \in B$ ,  $L_y$  is the left multiplication operator defined on  $A$ :  $L_y x = yx$ ,  $x \in A$ . We note that  $y \rightarrow L_y$  is a homomorphism of  $B$  into  $\mathcal{L}(A)$ . Finally, we denote by  $\mathcal{C}(A)$  the closure in  $\mathcal{L}(A)$  of  $\{L_x : x \in A\}$ .

Our characterization is as follows:

**THEOREM.** Let  $A$  be a proper  $H^*$ -algebra, with  $\hat{A} = l^2(\{B_s(H_\gamma), k_\gamma\})$ . For  $y \in l^\infty(\{B(H_\gamma)\})$ , define  $T_y$  on  $A$  by

$$(T_y x)^\wedge = L_y \hat{x}, \quad x \in A.$$

Then (i)  $y \rightarrow T_y$  is an isometric isomorphism of  $l^\infty(\{B(H_\gamma)\})$  and  $\mathcal{L}(A)$  and (ii) under this isomorphism,  $c_0(\{B_s(H_\gamma)\})$  corresponds to  $\mathcal{C}(A)$ .

The above characterization when  $A$  is a dual  $B^*$ -algebra is given in the proof of Theorem 3.1 of [7]. (Strictly speaking, the characterization in [7] is given for right centralizers.)

Before proceeding to the proof of the theorem, we establish the following lemma.

**LEMMA.** (Cf. [3, Theorem 18].) Let  $H$  be a Hilbert space. Then  $y \rightarrow L_y$  is an isometric isomorphism of  $B(H)$  and  $\mathcal{L}(B_s(H))$ .

**PROOF.** If  $y \in B(H)$ ,  $x \in B_s(H)$ , then  $\|yx\|_s \leq \|y\|_0 \|x\|_s$ , so  $y \rightarrow L_y$  is norm decreasing.

For  $\eta, \xi \in H$ , define the operator  $\eta \otimes \xi$  on  $H$  by  $\eta \otimes \xi(\mu) = \langle \mu, \xi \rangle \eta$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $H$ . Then  $\|\eta \otimes \xi\|_s = \|\eta\| \|\xi\|$ . Choose  $\xi \in H$  with  $\|\xi\| = 1$ .

Now, let  $T \in \mathcal{L}(B_s(H))$ . Define  $y$  on  $H$  by

$$y(\eta) = T(\eta \otimes \xi)(\xi), \quad \eta \in H.$$

Then  $y \in B(H)$  and  $\|y\|_0 \leq \|T\|$ . If  $\eta \in H$ ,  $z \in B_s(H)$ , then

$$\begin{aligned} L_y(z)(\eta) &= y(z(\eta)) = T(z(\eta) \otimes \xi)(\xi) \\ &= T(z(\eta \otimes \xi))(\xi) = T(z)(\eta \otimes \xi)(\xi) \\ &= T(z)(\eta), \end{aligned}$$

so that  $T=L_y$ . Q.E.D.

PROOF OF THEOREM. In view of the isomorphism  $a \rightarrow \hat{a}$ , it is sufficient to show that

- (i)'  $y \rightarrow L_y$  is an isometric isomorphism of  $l^\infty(\{B(H_\gamma)\})$  and  $\mathcal{L}(\hat{A})$ , and
  - (ii)' under this isomorphism  $c_0(\{B_c(H_\gamma)\})$  corresponds to  $\mathcal{C}(\hat{A})$ .
- (i)' First suppose that  $y \in l^\infty(\{B(H_\gamma)\})$ . If  $x \in \hat{A}$ , then

$$yx(\gamma) = y(\gamma)x(\gamma)$$

and

$$\|y(\gamma)x(\gamma)\|_s \leq \|y(\gamma)\|_0 \|x(\gamma)\|_s.$$

Hence  $\|yx\|_2^2 = \sum_\gamma k_\gamma^2 \|y(\gamma)x(\gamma)\|_s^2 \leq \|y\|_\infty^2 \|x\|_2^2$ . Thus  $\hat{A}$  is a left ideal in  $l^\infty(\{B(H_\gamma)\})$  and  $y \rightarrow L_y$  is a norm decreasing homomorphism of  $l^\infty(\{B(H_\gamma)\})$  into  $\mathcal{L}(\hat{A})$ .

It remains to show that  $y \rightarrow L_y$  is an isometry onto  $\mathcal{L}(\hat{A})$ . To this end, suppose that  $T \in \mathcal{L}(\hat{A})$ . For  $\gamma \in \Gamma$  let  $T_\gamma = T|_{\hat{A}_\gamma}$ . Since  $\hat{A}_\gamma^2$  is dense in  $\hat{A}_\gamma$  and  $T(\hat{A}_\gamma^2) \subset \hat{A}_\gamma$ , we have that  $T_\gamma \in \mathcal{L}(\hat{A}_\gamma)$ . Now  $\|x\|_2 = k_\gamma \|x(\gamma)\|_s$ ,  $x \in \hat{A}_\gamma$ . Thus  $T_\gamma$  induces an element  $\tilde{T}_\gamma \in \mathcal{L}(B_s(H_\gamma))$  given by  $\tilde{T}_\gamma(x(\gamma)) = (T_\gamma x)(\gamma)$ ,  $x \in \hat{A}_\gamma$ , and  $\|T_\gamma\| = \|\tilde{T}_\gamma\|$ .

By the lemma, there exists  $y(\gamma) \in B(H_\gamma)$  with  $\tilde{T}_\gamma = L_{y(\gamma)}$  and

$$\|y(\gamma)\|_0 = \|\tilde{T}_\gamma\| = \|T_\gamma\| \leq \|T\|, \quad \gamma \in \Gamma.$$

Thus  $y \in l^\infty(\{B(H_\gamma)\})$  and  $\|y\|_\infty \leq \|T\|$ .

If  $x \in \hat{A}$ ,  $\gamma \in \Gamma$ , then

$$(L_y x)(\gamma) = y(\gamma)x(\gamma) = \tilde{T}_\gamma(x(\gamma)) = (T_\gamma x)(\gamma) = (Tx)(\gamma),$$

so  $L_y = T$ .

(ii)' If  $x \in \hat{A}$ , then  $x \in c_0(\{B_s(H_\gamma)\}) \subset c_0(\{B_c(H_\gamma)\})$ , and  $c_0(\{B_c(H_\gamma)\})$  is closed in  $l^\infty(\{B(H_\gamma)\})$ . Now  $\mathcal{C}(\hat{A})$  is the closure of  $\{L_x : x \in \hat{A}\}$  in  $\mathcal{L}(\hat{A})$ , so every element of  $\mathcal{C}(\hat{A})$  corresponds to an element of  $c_0(\{B_c(H_\gamma)\})$ .

Conversely, if  $y \in c_0(\{B_c(H_\gamma)\})$ , we want to show  $L_y \in \mathcal{C}(\hat{A})$ . Since the finitely supported functions are dense in  $c_0(\{B_c(H_\gamma)\})$ , it is enough to show, for each  $\gamma$ , that  $L_y \in \mathcal{C}(\hat{A})$  when  $y(\gamma) \in B_c(H_\gamma)$  and  $y(\gamma') = 0$ ,  $\gamma' \neq \gamma$ . But this is equivalent to showing that  $L_{y(\gamma)} \in \mathcal{C}(B_s(H_\gamma))$  when  $y(\gamma) \in B_c(H_\gamma)$ , and this latter fact is true, using the lemma, since  $B_s(H_\gamma)$  is dense in  $B_c(H_\gamma)$ . Q.E.D.

**Conclusion.** We conclude with several remarks. In (I)–(IV) we assume that  $A$  is a proper  $H^*$ -algebra, with  $\hat{A} = l^2(\{B_s(H_\gamma), k_\gamma\})$ .

(I) When  $G$  is a compact group and  $A = L^2(G)$ , with convolution for multiplication, then each  $H_\gamma$  is finite dimensional,  $a \rightarrow \hat{a}$  is simply the Fourier transform, and  $k_\gamma = d_\gamma^{1/2}$ , where  $d_\gamma$  is the dimension of  $H_\gamma$ . As in [2], one calls a function  $y$  on  $\Gamma$ , with  $y(\gamma) \in B(H_\gamma)$ , a left  $(A, A)$  multiplier if  $y\hat{x} \in \hat{A}$ ,  $x \in A$ . In this case it is known that  $T \in \mathcal{L}(A)$  if and only if  $(Tx)^\hat{=} = y\hat{x}$  for some left  $(A, A)$  multiplier  $y$ , and that the left  $(A, A)$  multipliers coincide with  $l^\infty(\{B(H_\gamma)\})$  [2, Theorem 35.4].

(II) If each  $H_\gamma$  is finite dimensional, then  $\mathcal{C}(A)$  coincides with the set of compact left centralizers (cf. [1, Theorem 3]). Conversely, if each  $T \in \mathcal{C}(A)$  is compact, then each  $H_\gamma$  is finite dimensional by [5, Lemma 4].

(III) Let  $\mathcal{M}(A)$  denote the set of  $T \in \mathcal{L}(A)$  such that

$$T(xy) = xT(y) = T(x)y, \quad x, y \in A.$$

If  $T \in \mathcal{M}(A)$ ,  $\gamma \in \Gamma$ , then, in the notation of the above proof,  $\hat{T}_\gamma = L_{y(\gamma)}$ , where  $y(\gamma) \in B(H_\gamma)$  and  $y(\gamma)$  commutes with every element of  $B_s(H_\gamma)$ . Hence  $y(\gamma)$  is a multiple of the identity on  $H_\gamma$  by [8, Lemma 2.4.4]. Thus  $\mathcal{M}(A)$  corresponds to  $l^\infty(\Gamma)$  (which is [1, Theorem 2]), and  $\mathcal{M}(A) \cap \mathcal{C}(A)$  to  $c_0(\Gamma)$ .

(IV) Saworotnow and Friedell have defined the trace class of  $A$ ,  $\tau(A)$  [10]. A theorem of theirs [unpublished] states that  $\tau(A)$  is isometrically isomorphic to  $l^1(\{B_i(H_\gamma), k_\gamma\})$ , where  $B_i(H_\gamma)$  denotes the algebra of operators of trace class, endowed with the trace class norm.

For a Banach space  $X$ , let  $X^*$  denote its dual space. Our characterization enables us to give an alternate proof of the theorems of Saworotnow [9] that  $\mathcal{C}(A)^*$  is isometrically isomorphic to  $\tau(A)$  and  $\tau(A)^*$  is isometrically isomorphic to  $\mathcal{L}(A)$  (cf. [7, proof of Theorem 3.1]):

Since  $B_c(H_\gamma)^*$  is isometrically isomorphic to  $B_i(H_\gamma)$  and  $B_i(H_\gamma)^*$  is isometrically isomorphic to  $B(H_\gamma)$  [11], one can show that  $c_0(\{B_c(H_\gamma)\})^*$  is isometrically isomorphic to  $l^1(\{B_i(H_\gamma), k_\gamma\})$  and that  $l^1(\{B_i(H_\gamma), k_\gamma\})^*$  is isometrically isomorphic to  $l^\infty(\{B(H_\gamma)\})$ . Using the identification of  $\mathcal{C}(A)$  with  $c_0(\{B_c(H_\gamma)\})$ ,  $\tau(A)$  with  $l^1(\{B_i(H_\gamma), k_\gamma\})$ , and  $\mathcal{L}(A)$  with  $l^\infty(\{B(H_\gamma)\})$ , one thus obtains Saworotnow's results.

(V) The lemma admits the following generalization: *Suppose  $X$  is a Banach space and  $I$  is a left ideal of  $B(X)$  which is a Banach algebra in some norm dominating the operator norm. Then  $y \rightarrow L_y$  is a bicontinuous isomorphism of  $B(X)$  onto  $\mathcal{L}(I)$ .*

One first notes that, for some  $f \in X^*$  with  $\|f\|=1$ ,  $I$  must contain the minimal left ideal  $J = \{\eta \otimes f : \eta \in X\}$ . Now  $\|\eta \otimes f\|_I \geq \|\eta \otimes f\|_0 = \|\eta\|$ , so it follows that  $\eta \otimes f \mapsto \eta$  gives a linear homeomorphism of  $J$  and  $X$ . Choose  $\xi \in X$  with  $f(\xi) = 1$ . Then  $y(\eta) = T(\eta \otimes f)(\xi)$ ,  $\eta \in X$ , defines a

bounded linear operator on  $X$ , and, as in the proof of the lemma, one shows that  $T=L_y$ . The mapping  $y \rightarrow L_y$  is then a continuous isomorphism of  $B(X)$  onto  $\mathcal{L}(I)$ , and hence bicontinuous.

(VI) Suppose that  $A$  is a semisimple annihilator Banach algebra, with  $\{A_\gamma\}_{\gamma \in \Gamma}$  its collection of minimal closed two-sided ideals. If  $X_\gamma$  is a minimal left ideal of  $A_\gamma$ , let  $\hat{A}_\gamma$  denote the image of  $A_\gamma$  in  $B(X_\gamma)$  under the left regular representation. The norm in  $\hat{A}_\gamma$  transported from  $A_\gamma$  dominates the operator norm. If  $\hat{A}_\gamma$  is a left ideal in  $B(X_\gamma)$ , then, using (V), one has that  $\mathcal{L}(A_\gamma)$  is bicontinuously isomorphic to  $B(X_\gamma)$ .

If, in addition,  $A$  can be represented as  $l^p(\{A_\gamma\})$ ,  $1 \leq p < \infty$ , or  $c_0(\{A_\gamma\})$ , and each  $\hat{A}_\gamma$  is a norm left ideal in  $B(X_\gamma)$  (i.e., the norm in  $\hat{A}_\gamma$ ,  $\|\cdot\|_\gamma$ , is a cross norm and  $\|y\hat{x}\|_\gamma \leq \|y\|_0 \|\hat{x}\|_\gamma$  for  $y \in B(X_\gamma)$ ,  $x \in A_\gamma$ ), then one can show that  $\mathcal{L}(A)$  is isometrically isomorphic to  $l^\infty(\{B(X_\gamma)\})$ . The proof is virtually the same as that of the theorem.

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