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LEFT CENTRALIZERS OF AN H^*-ALGEBRA

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ABSTRACT. An explicit characterization is given of the left centralizers of a proper H^*-algebra A. Each left centralizer is seen to correspond to a bounded family of bounded operators, where each operator acts on a Hilbert space associated with a minimal-closed two-sided ideal of A.

Introduction. Let A be a semisimple Banach algebra. As in **[3]**, we call a linear operator T on A a *left centralizer* if

$$T(xy) = T(x)y, \qquad x, y \in A.$$

In this note we give an explicit characterization of the left centralizers on A when A is a proper H^*-algebra. Centralizers on H^*-algebras have been considered in **[1]**, **[6]**, and **[9]**. The same characterization holds when A is a dual B^*-algebra, and has been given by Malviya and Tomiuk in **[7]**. Our proof is similar to that in **[7]**. We include most details for the sake of completeness.

Use will be made of the structure theory of H^*-algebras (see e.g. **[8]**), which we shall review here briefly, after introducing some notation.

Given a family of Banach algebras, $\{A_\gamma\}_{\gamma\in\Gamma}$, and numbers $k_\gamma \geqq 1$, we denote by $l^p(\{A_\gamma, k_\gamma\})$, $1 \leqq p < \infty$, the set of functions x on Γ with $x(\gamma) \in A_\gamma$ and

$$\|x\|_p = \left(\sum_\gamma k_\gamma^p \|x(\gamma)\|^p\right)^{1/p} < \infty.$$

We denote by $l^\infty(\{A_\gamma\})$ the set of functions x on Γ with $x(\gamma) \in A_\gamma$ and

$$\|x\|_\infty = \sup_\gamma \|x(\gamma)\| < \infty.$$

With the usual operations for functions and the norm $\|x\|_p$, the above sets become Banach algebras. We denote by $c_0(\{A_\gamma\})$ the closed subalgebra of $l^\infty(\{A_\gamma\})$ consisting of those functions x for which $\{\gamma : \|x(\gamma)\| \geqq \varepsilon\}$ is finite for all $\varepsilon > 0$.

Given a Hilbert space H, $B(H)$ denotes the algebra of bounded linear operators on H, endowed with the operator norm, $\| \ \|_0$; $B_c(H)$ denotes

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the closed two-sided ideal of compact operators. We denote by $B_s(H)$ the two-sided ideal of Hilbert-Schmidt operators, endowed with the Hilbert-Schmidt norm, $\| \ \|_s$. With this latter norm, $B_s(H)$ is a Banach algebra (see [8] or [11]).

If A is a proper H^*-algebra, let $\{A_\gamma\}_{\gamma\in\Gamma}$ denote its collection of minimal-closed two-sided ideals. For each γ, let H_γ be some minimal left ideal of A_γ. Then, under the left regular representation, A_γ is isomorphic to $B_s(H_\gamma)$, and there exist $k_\gamma \geqq 1$ such that *A is isometrically isomorphic to $l^2(\{B_s(H_\gamma), k_\gamma\})$. We denote this isomorphism by $a \to \hat{a}$.* For $S \subset A$, let $\hat{S} = \{\hat{a} : a \in S\}$. Then $\hat{A}_\gamma = \{x \in l^2(\{B_s(H_\beta), k_\beta\}) : x(\beta) = 0, \beta \neq \gamma\}$.

The main result. If A is a semisimple Banach algebra, we denote the left centralizers on A by $\mathscr{L}(A)$. A theorem of Johnson and Sinclair states that any left centralizer on A is continuous [4]. When endowed with the operator norm, $\mathscr{L}(A)$ is a Banach algebra. We denote this norm simply by $\| \ \|$. When A is a left ideal in a Banach algebra B, then, for $y \in B$, L_y is the left multiplication operator defined on A: $L_y x = yx$, $x \in A$. We note that $y \to L_y$ is a homomorphism of B into $\mathscr{L}(A)$. Finally, we denote by $\mathscr{C}(A)$ the closure in $\mathscr{L}(A)$ of $\{L_x : x \in A\}$.

Our characterization is as follows:

THEOREM. *Let A be a proper H^*-algebra, with $\hat{A} = l^2(\{B_s(H_\gamma), k_\gamma\})$. For $y \in l^\infty(\{B(H_\gamma)\})$, define T_y on A by*

$$(T_y x)^\wedge = L_y \hat{x}, \qquad x \in A.$$

Then (i) *$y \to T_y$ is an isometric isomorphism of $l^\infty(\{B(H_\gamma)\})$ and $\mathscr{L}(A)$ and* (ii) *under this isomorphism, $c_0(\{B_c(H_\gamma)\})$ corresponds to $\mathscr{C}(A)$.*

The above characterization when A is a dual B^*-algebra is given in the proof of Theorem 3.1 of [7]. (Strictly speaking, the characterization in [7] is given for right centralizers.)

Before proceeding to the proof of the theorem, we establish the following lemma.

LEMMA. (*Cf.* [3, *Theorem* 18].) *Let H be a Hilbert space. Then $y \to L_y$ is an isometric isomorphism of $B(H)$ and $\mathscr{L}(B_s(H))$.*

PROOF. If $y \in B(H)$, $x \in B_s(H)$, then $\|yx\|_s \leqq \|y\|_0 \|x\|_s$, so $y \to L_y$ is norm decreasing.

For η, $\xi \in H$, define the operator $\eta \otimes \xi$ on H by $\eta \otimes \xi(\mu) = \langle \mu, \xi \rangle \eta$, where $\langle \ , \ \rangle$ denotes the inner product in H. Then $\|\eta \otimes \xi\|_s = \|\eta\| \, \|\xi\|$. Choose $\xi \in H$ with $\|\xi\| = 1$.

Now, let $T \in \mathscr{L}(B_s(H))$. Define y on H by

$$y(\eta) = T(\eta \otimes \xi)(\xi), \qquad \eta \in H.$$

Then $y \in B(H)$ and $\|y\|_0 \leqq \|T\|$. If $\eta \in H$, $z \in B_s(H)$, then

$$\begin{aligned} L_y(z)(\eta) &= y(z(\eta)) = T(z(\eta) \otimes \xi)(\xi) \\ &= T(z(\eta \otimes \xi))(\xi) = T(z)(\eta \otimes \xi)(\xi) \\ &= T(z)(\eta), \end{aligned}$$

so that $T = L_y$. Q.E.D.

PROOF OF THEOREM. In view of the isomorphism $a \to \hat{a}$, it is sufficient to show that

(i)′ $y \to L_y$ is an isometric isomorphism of $l^\infty(\{B(H_\gamma)\})$ and $\mathscr{L}(\hat{A})$, and

(ii)′ under this isomorphism $c_0(\{B_c(H_\gamma)\})$ corresponds to $\mathscr{C}(\hat{A})$.

(i)′ First suppose that $y \in l^\infty(\{B(H_\gamma)\})$. If $x \in \hat{A}$, then

$$yx(\gamma) = y(\gamma)x(\gamma)$$

and

$$\|y(\gamma)x(\gamma)\|_s \leqq \|y(\gamma)\|_0 \, \|x(\gamma)\|_s.$$

Hence $\|yx\|_2^2 = \sum_\gamma k_\gamma^2 \|y(\gamma)x(\gamma)\|_s^2 \leqq \|y\|_\infty^2 \|x\|_2^2$. Thus $\hat{A}$ is a left ideal in $l^\infty(\{B(H_\gamma)\})$ and $y \to L_y$ is a norm decreasing homomorphism of $l^\infty(\{B(H_\gamma)\})$ into $\mathscr{L}(\hat{A})$.

It remains to show that $y \to L_y$ is an isometry onto $\mathscr{L}(\hat{A})$. To this end, suppose that $T \in \mathscr{L}(\hat{A})$. For $\gamma \in \Gamma$ let $T_\gamma = T|\hat{A}_\gamma$. Since $\hat{A}_\gamma^2$ is dense in $\hat{A}_\gamma$ and $T(\hat{A}_\gamma^2) \subset \hat{A}_\gamma$, we have that $T_\gamma \in \mathscr{L}(\hat{A}_\gamma)$. Now $\|x\|_2 = k_\gamma \|x(\gamma)\|_s$, $x \in \hat{A}_\gamma$. Thus T_γ induces an element $\tilde{T}_\gamma \in \mathscr{L}(B_s(H_\gamma))$ given by $\tilde{T}_\gamma(x(\gamma)) = (T_\gamma x)(\gamma)$, $x \in \hat{A}_\gamma$, and $\|T_\gamma\| = \|\tilde{T}_\gamma\|$.

By the lemma, there exists $y(\gamma) \in B(H_\gamma)$ with $\tilde{T}_\gamma = L_{y(\gamma)}$ and

$$\|y(\gamma)\|_0 = \|\tilde{T}_\gamma\| = \|T_\gamma\| \leqq \|T\|, \qquad \gamma \in \Gamma.$$

Thus $y \in l^\infty(\{B(H_\gamma)\})$ and $\|y\|_\infty \leqq \|T\|$.

If $x \in \hat{A}$, $\gamma \in \Gamma$, then

$$(L_y x)(\gamma) = y(\gamma)x(\gamma) = \tilde{T}_\gamma(x(\gamma)) = (T_\gamma x)(\gamma) = (Tx)(\gamma),$$

so $L_y = T$.

(ii)′ If $x \in \hat{A}$, then $x \in c_0(\{B_s(H_\gamma)\}) \subset c_0(\{B_c(H_\gamma)\})$, and $c_0(\{B_c(H_\gamma)\})$ is closed in $l^\infty(\{B(H_\gamma)\})$. Now $\mathscr{C}(\hat{A})$ is the closure of $\{L_x : x \in \hat{A}\}$ in $\mathscr{L}(\hat{A})$, so every element of $\mathscr{C}(\hat{A})$ corresponds to an element of $c_0(\{B_c(H_\gamma)\})$.

Conversely, if $y \in c_0(\{B_c(H_\gamma)\})$, we want to show $L_y \in \mathscr{C}(\hat{A})$. Since the finitely supported functions are dense in $c_0(\{B_c(H_\gamma)\})$, it is enough to show, for each γ, that $L_y \in \mathscr{C}(\hat{A})$ when $y(\gamma) \in B_c(H_\gamma)$ and $y(\gamma') = 0$, $\gamma' \neq \gamma$. But this is equivalent to showing that $L_{y(\gamma)} \in \mathscr{C}(B_s(H_\gamma))$ when $y(\gamma) \in B_c(H_\gamma)$, and this latter fact is true, using the lemma, since $B_s(H_\gamma)$ is dense in $B_c(H_\gamma)$. Q.E.D.

Conclusion. We conclude with several remarks. *In* (I)–(IV) *we assume that A is a proper H^*-algebra, with $\hat{A}=l^2(\{B_s(H_\gamma), k_\gamma\})$.*

(I) When G is a compact group and $A=L^2(G)$, with convolution for multiplication, then each H_γ is finite dimensional, $a\to\hat{a}$ is simply the Fourier transform, and $k_\gamma=d_\gamma^{1/2}$, where d_γ is the dimension of H_γ. As in [**2**], one calls a function y on Γ, with $y(\gamma)\in B(H_\gamma)$, a left (A, A) multiplier if $y\hat{x}\in\hat{A}$, $x\in A$. In this case it is known that $T\in\mathscr{L}(A)$ if and only if $(Tx)\hat{}=y\hat{x}$ for some left (A, A) multiplier y, and that the left (A, A) multipliers coincide with $l^\infty(\{B(H_\gamma)\})$ [**2**, Theorem 35.4].

(II) If each H_γ is finite dimensional, then $\mathscr{C}(A)$ coincides with the set of compact left centralizers (cf. [**1**, Theorem 3]). Conversely, if each $T\in\mathscr{C}(A)$ is compact, then each H_γ is finite dimensional by [**5**, Lemma 4].

(III) Let $\mathscr{M}(A)$ denote the set of $T\in\mathscr{L}(A)$ such that

$$T(xy) = xT(y) = T(x)y, \qquad x, y\in A.$$

If $T\in\mathscr{M}(A)$, $\gamma\in\Gamma$, then, in the notation of the above proof, $\tilde{T}_\gamma=L_{y(\gamma)}$, where $y(\gamma)\in B(H_\gamma)$ and $y(\gamma)$ commutes with every element of $B_s(H_\gamma)$. Hence $y(\gamma)$ is a multiple of the identity on H_γ by [**8**, Lemma 2.4.4]. Thus $\mathscr{M}(A)$ corresponds to $l^\infty(\Gamma)$ (which is [**1**, Theorem 2]), and $\mathscr{M}(A)\cap\mathscr{C}(A)$ to $c_0(\Gamma)$.

(IV) Saworotnow and Friedell have defined the trace class of A, $\tau(A)$ [**10**]. A theorem of theirs [unpublished] states that $\tau(A)$ is isometrically isomorphic to $l^1(\{B_t(H_\gamma), k_\gamma\})$, where $B_t(H_\gamma)$ denotes the algebra of operators of trace class, endowed with the trace class norm.

For a Banach space X, let X^* denote its dual space. Our characterization enables us to give an alternate proof of the theorems of Saworotnow [**9**] that $\mathscr{C}(A)^*$ is isometrically isomorphic to $\tau(A)$ and $\tau(A)^*$ is isometrically isomorphic to $\mathscr{L}(A)$ (cf. [**7**, proof of Theorem 3.1]):

Since $B_c(H_\gamma)^*$ is isometrically isomorphic to $B_t(H_\gamma)$ and $B_t(H_\gamma)^*$ is isometrically isomorphic to $B(H_\gamma)$ [**11**], one can show that $c_0(\{B_c(H_\gamma)\})^*$ is isometrically isomorphic to $l^1(\{B_t(H_\gamma), k_\gamma\})$ and that $l^1(\{B_t(H_\gamma), k_\gamma\})^*$ is isometrically isomorphic to $l^\infty(\{B(H_\gamma)\})$. Using the identification of $\mathscr{C}(A)$ with $c_0(\{B_c(H_\gamma)\})$, $\tau(A)$ with $l^1(\{B_t(H_\gamma), k_\gamma\})$, and $\mathscr{L}(A)$ with $l^\infty(\{B(H_\gamma)\})$, one thus obtains Saworotnow's results.

(V) The lemma admits the following generalization: *Suppose X is a Banach space and I is a left ideal of $B(X)$ which is a Banach algebra in some norm dominating the operator norm. Then $y\to L_y$ is a bicontinuous isomorphism of $B(X)$ onto $\mathscr{L}(I)$.*

One first notes that, for some $f\in X^*$ with $\|f\|=1$, I must contain the minimal left ideal $J=\{\eta\otimes f: \eta\in X\}$. Now $\|\eta\otimes f\|_I\geqq\|\eta\otimes f\|_0=\|\eta\|$, so it follows that $\eta\otimes f\leftrightarrow\eta$ gives a linear homeomorphism of J and X. Choose $\xi\in X$ with $f(\xi)=1$. Then $y(\eta)=T(\eta\otimes f)(\xi)$, $\eta\in X$, defines a bounded linear operator on X, and, as in the proof of the lemma, one shows that $T=L_y$. The mapping $y\to L_y$ is then a continuous isomorphism of $B(X)$ onto $\mathscr{L}(I)$, and hence bicontinuous.

(VI) Suppose that A is a semisimple annihilator Banach algebra, with $\{A_\gamma\}_{\gamma\in\Gamma}$ its collection of minimal closed two-sided ideals. If X_γ is a minimal left ideal of A_γ, let $\hat{A}_\gamma$ denote the image of A_γ in $B(X_\gamma)$ under the left regular representation. The norm in $\hat{A}_\gamma$ transported from A_γ dominates the operator norm. If $\hat{A}_\gamma$ is a left ideal in $B(X_\gamma)$, then, using (V), one has that $\mathscr{L}(A_\gamma)$ is bicontinuously isomorphic to $B(X_\gamma)$.

If, in addition, A can be represented as $l^p(\{A_\gamma\})$, $1\leqq p<\infty$, or $c_0(\{A_\gamma\})$, and each $\hat{A}_\gamma$ is a norm left ideal in $B(X_\gamma)$ (i.e., the norm in $\hat{A}_\gamma$, $\|\ \|_\gamma$, is a cross norm and $\|y\hat{x}\|_\gamma\leqq\|y\|_0\|\hat{x}\|_\gamma$ for $y\in B(X_\gamma)$, $x\in A_\gamma$), then one can show that $\mathscr{L}(A)$ is isometrically isomorphic to $l^\infty(\{B(X_\gamma)\})$. The proof is virtually the same as that of the theorem.

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