PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, March 1974

THE ROLE OF ZERO SETS IN THE SPECTRA OF HYPONORMAL OPERATORS[1]

C. R. PUTNAM

ABSTRACT. A compact set in the complex plane is the spectrum of a completely hyponormal operator if and only if the set has positive density.

1. **Introduction.** A bounded operator T on a Hilbert space is said to be hyponormal if $T^*T-TT^*\geqq 0$, and completely hyponormal if there exists no nontrivial reducing space on which it is normal. It was shown in [**5**] that if T is completely hyponormal and if its spectrum, $\mathrm{sp}(T)$, intersects an open disk in a nonempty set, then the (planar) measure of the intersection is positive. It will be shown below that, conversely, if S is any compact set of the plane having positive density, in the sense that

$$S \cap N \neq \varnothing \Rightarrow \mathrm{meas}_2(S \cap N) > 0, \qquad N \text{ any open disk,} \tag{1.1}$$

then there exists a completely hyponormal operator T for which $S=\mathrm{sp}(T)$. Thus,

THEOREM 1. *A necessary and sufficient condition that a compact set S be the spectrum of a completely hyponormal operator is that* (1.1) *holds.*

Consequently, sets of measure 0 play the same role for hyponormal operators as the nowhere dense compact sets X for which $C(X)=R(X)$ play for subnormal operators; see [**1**].

As noted above, the necessity part of Theorem 1 was proved in [**5**], so that only the proof of the sufficiency remains. This will be given in §4 as a consequence of Theorem 2 stated next, the proof of which will be given in §3.

THEOREM 2. *Let S be a subset of the plane satisfying*

$$S \text{ is compact} \quad \text{and} \quad \mathrm{meas}_2(S) > 0. \tag{1.2}$$

Then there exists a completely hyponormal operator M for which M^*M-MM^* has rank 1 *and* $\mathrm{sp}(M)\subset S$.

Received by the editors June 20, 1972 and, in revised form, March 26, 1973.
AMS (MOS) subject classifications (1970). Primary 47B20; Secondary 47A10.
Key words and phrases. Hyponormal operator, subnormal operator, spectrum.
[1] This work was supported by a National Science Foundation research grant 6460-50-1395515-31-74.

A certain result, given below as a lemma and the validity of which remained problematic in [**6**], will be needed. (The author has since learned that the lemma follows from some results of R. O. Davies [**4**]; see §2 below.) Some terminology will first be given. Let α denote any closed set of real numbers and let S_α denote the infinite closed strip $S_\alpha=\{z:\mathrm{Re}(z)\in\alpha\}$. For each t, $0\leqq t<2\pi$, let $S(t,\alpha)=e^{it}S_\alpha$, the set S_α rotated about the origin by the angle t.

LEMMA. *Let S be a set of the plane for which* (1.2) *holds. Then there exists a (finite or infinite) sequence $t_1, t_2, \cdots$, $0\leqq t_n<2\pi$, and a sequence of compact sets $\alpha(t_n)$ on the real line for which the closed set*

$$A=\bigcap_n S(t_n,\alpha(t_n)) \tag{1.3}$$

satisfies

$$A\subset S \quad \textit{and} \quad \mathrm{meas}_2(A)>0. \tag{1.4}$$

In case each $\alpha(t_n)$ is a closed interval, the set A is convex (and closed). Thus the lemma states that any compact (hence, any measurable) set of positive planar measure contains a "generalized" compact convex subset of positive measure. It may be noted that the corresponding assertion concerning the existence of generalized rectangles is false; see [**3**].

2. **Proof of the lemma.** Choose an open rectangle R with sides parallel to the coordinate axes and so that $S\subset R$. By choosing appropriate open vertical half-planes $\{z:-\infty<\mathrm{Re}(z)<a\}$ and $\{z:b<\mathrm{Re}(z)<\infty\}$ and a corresponding pair of horizontal half-planes one can "erase" the set $C-R^-$. (Here C denotes the complex plane and R^- denotes the closure of R.) Thus one may restrict attention to the set $R-S$, the complement of S with respect to R. It is clearly sufficient to show that there exists a sequence of open intervals (a_n, b_n), where $-\infty<a_n<b_n<\infty$, and real numbers t_n, where $0\leqq t_n<2\pi$, together with corresponding open strips

$$S_n=e^{it_n}\{z:a_n<\mathrm{Re}(z)<b_n\}, \tag{2.1}$$

for which

$$(R-S)\subset\left(\bigcup_n S_n\right) \quad \text{and} \quad \mathrm{meas}_2\left(S-\left(\bigcup_n S_n\right)\right)>0. \tag{2.2}$$

The author is indebted to Professor R. O. Davies for informing him that the above desired result, hence the lemma, is essentially contained in Davies' paper [**4**]. In fact, it was shown there [*loc. cit.*, p. 220] that if E is any set of finite planar measure then there exists a planar measurable set L composed of straight lines and satisfying $E\subset L$ and $\mathrm{meas}_2(L-E)=0$. The proof of this last result depends on Lemma 6 of [**4**, p. 220]. A special case of that lemma will be needed in the present paper and will be formulated as the following:

LEMMA (*) [R. O. DAVIES]. *Let R be an open rectangle $ABB'A'$ with horizontal sides AB and $A'B'$, and let E be any planar measurable set contained in R. Then, for any $\varepsilon>0$, there exists a sequence of open parallelograms $\{P_n\}$, $n=1, 2, \cdots$, contained in R and with two sides on AB and $A'B'$ for which*

$$E \subset \bigcup_n P_n \quad \textit{and} \quad \mathrm{meas}_2\left(\bigcup_n P_n - E\right) < \varepsilon. \tag{2.3}$$

First, it may be noted that, in Lemma 6 of [4], the corresponding sets R and P_n are all closed parallelograms. The formulation of Lemma (*) in which R is an open rectangle and the P_n are open parallelograms is a trivial modification and will be more convenient for our application below.

Next, let the rectangle R and the set E of Lemma (*) be identified with the sets R and $R-S$ occurring in the beginning of this section. Further, for a given $\varepsilon>0$, let each parallelogram P_n be extended to an open strip S_n by simply removing its sides lying on AB and $A'B'$ and extending the other two sides indefinitely in both directions. If $B=\bigcup_n S_n$, then, by (2.3),

$$R - S \subset B \quad \text{and} \quad \mathrm{meas}_2(B \cap S) = \mathrm{meas}_2\left(\bigcup_n P_n - (R - S)\right) < \varepsilon.$$

Since $\mathrm{meas}_2(S)>0$ and since $S=(S-B)\cup(S\cap B)$, relation (2.2) now follows by choosing ε to be less than $\mathrm{meas}_2(S)$.

3. **Proof of Theorem 2.** Since M can be replaced by cM where $c=\mathrm{const}>0$, it can be supposed that S is a subset of the closed unit disk. Let T denote the unilateral shift operator, so that $\mathrm{sp}(T)$ is this disk and let A satisfy (1.3) and (1.4) of the lemma. Next, note that T and A may be identified with the corresponding symbols in Theorem 2 of [6, p. 702]. Actually, it was assumed there that $\mathrm{sp}(T)\cap A$ had positive density, a condition not now assumed. Nevertheless, the proof given there [*loc. cit.*, p. 706, bottom] shows that there exists an orthogonal projection P for which $T_P=PTP$ is hyponormal; in fact, $T_P^*T_P-T_PT_P^*=P(T^*T-TT^*)P$. Further, $\mathrm{sp}(T_P)\subset(\mathrm{sp}(T)\cap A)$ and $(\mathrm{sp}(T)\cap A)-\mathrm{sp}(T_P)$ is a set of measure 0. (Here, T_P as well as T_Q below are regarded as operators on the corresponding projections of the original Hilbert space.) Since $\mathrm{sp}(T)\cap S=S$, it is clear from (1.4) that $\mathrm{sp}(T_P)$ differs from A by a set of measure 0. If T_P is not already completely hyponormal one can choose (cf. [6, p. 707]) a projection $Q\leqq P$ so that $M=T_Q=QTQ$ is completely hyponormal, M^*M-MM^* has rank 1, and $\mathrm{sp}(M)$ differs from $\mathrm{sp}(T_P)$,

hence from A, by a set of measure 0. But $\mathrm{sp}(M)$ must have positive density (necessity part of Theorem 1) and hence $\mathrm{sp}(M) \subset A$ $(\subset S)$. This proves Theorem 2.

4. **Proof of sufficiency in Theorem 1.** This follows easily by taking direct sums of operators of the type M in Theorem 2. Thus, since S has positive density, there exists a sequence of points $\{z_n\}$ belonging to S and which are dense in S, and open disks $N_n = \{z : |z - z_n| < r_n\}$ such that each set $S \cap N_n$ has positive measure. Then choose M_n as in Theorem 2 so that M_n is completely hyponormal, $M_n^* M_n - M_n M_n^*$ has rank 1 and $\mathrm{sp}(M_n) \subset (S \cap \bar{N_N})$. Then $T = \sum \oplus M_n$ is completely hyponormal and $\mathrm{sp}(T)$ is the closure of the set $\bigcup_n \mathrm{sp}(M_n)$; see [**6**, p. 703] for a similar argument. Since the $\{z_n\}$ are dense in S it is clear that $\mathrm{sp}(T) = S$.

ADDED IN PROOF. It has recently been shown by R. W. Carey and J. D. Pincus (*Construction of seminormal operators with prescribed mosaic* (Preprint)), by other methods, that the operator occurring in the sufficiency part of Theorem 1 above can be chosen so as to have a one-dimensional self-commutator.

REFERENCES

1. K. F. Clancey and C. R. Putnam, *The local spectral behavior of completely subnormal operators*, Trans. Amer. Math. Soc. **163** (1972), 239–244. MR **45** #934.

2. ———, *The spectra of hyponormal integral operators*, Comm. Math. Helv. **46** (1971), 451–456.

3. R. Darst and C. Goffman, *A Borel set which contains no rectangles*, Amer. Math. Monthly **77** (1970), 728–729. MR **41** #8615.

4. R. O. Davies, *On accessibility of plane sets and differentiation of functions of two real variables*, Proc. Cambridge Philos. Soc. **48** (1952), Part 2, 215–232. MR **13**, 635.

5. C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. **28** (1971), 473–477.

6. ———, *The spectra of completely hyponormal operators*, Amer. J. Math. **93** (1971), 699–708. MR **43** #6757.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, LAFAYETTE, INDIANA 47907