

SWITCHING SETS IN $PG(3, q)$

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ABSTRACT. In this note, we are mainly concerned with partial spreads U, V of $PG(3, q)$ which cover the same points and have no line in common. Setting $|U|=|V|=t$, we show that if $t > q+1$ then $t \geq \max(q+2, 2q-2)$. Certain applications of this result to $(0, 1)$ matrices and to translation planes are then discussed.

1. Summary. Our purpose is to describe a new result (Theorem 3) on the size of replaceable partial spreads in $\Sigma = PG(3, q)$, the 3 dimensional projective space over the finite field of order $q = p^s$ where p is a prime and s is a positive integer. The bound we obtain is "best possible" for q odd. It also represents somewhat of a breakthrough in the combinatorial theory of finite translation planes since, when applied to certain classes of such planes, it leads to an improvement of a well-known bound due to R. H. Bruck [1] (see Theorem 4). Furthermore, our result has a purely combinatorial interpretation in terms of $(0, 1)$ matrices; this is discussed in §5. Although we are dealing with very "modern" questions, our main tool is a classical theorem in solid geometry, namely the regulus theorem discussed below. By way of an example we shall also come across another hardy perennial—the Schläfli double-six configuration. The proof in outline of the main result is a pleasant mixture of geometry and combinatorics. In order to exhibit this, *we have attempted to keep this note accessible and self-contained*. Thus we discuss below some geometrical background. We should mention that Theorem 3 can also be stated in purely affine or vector space language. How it can be proved, however, without reverting to the projective situation is far from clear.

2. Background in Σ . By a *partial spread* U of $\Sigma = PG(3, q)$ we mean any nonempty family of pairwise skew lines of Σ . If every point P of Σ lies on some line of U we then refer to U as a full spread or, simply, a *spread* of Σ . Two partial spreads U, V of Σ are said to *cover the same points* provided the following holds: A point P of Σ lies on a line of U if and only if P lies on a line of V . A *transversal* of the partial spread

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X is a line of Σ meeting each line of X in exactly one point. Our main tool is the following result (see [3, p. 221]).

THEOREM 1 (REGULUS THEOREM). *Let a, b, c be 3 distinct and pairwise skew lines of Σ , and let a', b', c' be 3 distinct transversals of the partial spread $\{a, b, c\}$. Then each transversal of $\{a, b, c\}$ intersects each transversal of $\{a', b', c'\}$.*

This theorem has some important consequences. By a *regulus* we mean the partial spread consisting of all the transversals to 3 distinct and pairwise skew lines. From Theorem 1 we can show that 3 skew lines are contained in a *unique* regulus R . Moreover there is then also determined another (unique) regulus R' having no lines in common with R but such that R and R' cover the same points. $R'(R)$ is called the *opposite regulus* of $R(R')$ respectively. It follows also that a line which intersects 3 or more lines of a regulus X intersects all lines of X and is in fact a line of the opposite regulus X' . We note that a regulus contains exactly $q+1$ lines. Next, let U, V be partial spreads, each containing exactly 6 distinct lines with $U=\{u_i\}, V=\{v_i\}, 1 \leq i \leq 6$. Suppose that:

- (1) The line u_k is skew to $v_k, 1 \leq k \leq 6$.
- (2) Each u_i intersects each v_j provided $i \neq j, 1 \leq i, j \leq 6$.

We then refer to two such families U, V of 6 pairwise skew lines as a *double-six configuration*. We refer to Hirschfeld [5] for a proof of the following.

THEOREM 2. *A double-six configuration exists in $PG(3, 4)$.*

REMARK. We can think of Σ as being based on a 4-dimensional vector space $V=V_4(q)$, and analogous definitions for partial spread, spread, etc., can be phrased in terms of the subspaces of V .

3. The main result. We make repeated use of Theorem 1 and its consequences in proving the following.

THEOREM 3. *Let U and V be partial spreads of $\Sigma=PG(3, q)$ which cover the same points. Assume that U and V have no lines in common. Then we have the following:*

- (1) $|U|=|V|=t$, say.
- (2) $t \geq q+1$, and equality holds if and only if U is a regulus and V is its opposite regulus.
- (3) If $t > q+1$ then $t \geq \max(q+2, 2q-2)$.
- (4) If $q > 3$ is odd, the bound above in (3) is best possible being attained by constructions due to D. A. Foulser in [4].
- (5) If $q=2, |U| \geq 4$ and there exist examples with $|U|=4$.
- (6) If $q=4, t \geq 6$ and if $t=6$ then U, V must form a double-six configuration. Moreover, from Theorem 2, such a configuration does exist.

OUTLINE OF PROOF. Parts (1) and (2) are immediate from the definitions. Let us concentrate on part (3). There are two cases to consider. The first case is when any 3 lines of U have at most 3 transversals in V , and, at the same time, each 3 lines of V have at most 3 transversals in U . In the second case we can assume that some 3 lines of U (or V), say the lines u_1, u_2, u_3 of U , have at least 4 transversals v_1, v_2, v_3, v_4 in V . One can dispose of the first case by an ad hoc argument which is outlined in the appendix (§6). We focus on the second case. Let there be exactly β lines $v_1, v_2, v_3, v_4, \dots, v_\beta$ in V which are transversals to $\{u_1, u_2, u_3\}$. Since a regulus contains exactly $q+1$ lines, we have $4 \leq \beta \leq q+1$. Suppose there are exactly α transversals $u_1, u_2, u_3, \dots, u_\alpha$ to the set $\{v_1, v_2, v_3\}$. Actually, by the regulus theorem, each line u_i will meet each line v_j , $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$. Every point on the β lines v_j , $1 \leq j \leq \beta$, must be covered by a line of U . Each such line v_j has exactly $q+1$ points, and the α lines $u_1, u_2, u_3, \dots, u_\alpha$ account for just α of these points. Also, by the regulus theorem, a line x of U meeting more than 2 of these lines v_j meets all of them (in particular, x then meets v_1, v_2, v_3). Thus we obtain

$$|U| \geq \alpha + \frac{1}{2}\beta(q+1-\alpha).$$

Similarly

$$|V| \geq \beta + \frac{1}{2}\alpha(q+1-\beta).$$

For the sake of argument we can take $\alpha \leq \beta$. By assumption $\beta \geq 4$. Thus certainly $\beta > 2$. Therefore $|U| \geq \beta + \frac{1}{2}\beta(q+1-\beta)$. An examination of this quadratic, again using the fact that $\beta \geq 4$, shows that we are done unless $\beta \geq q$. Since the number of lines in a regulus is $q+1$, we now have $q \leq \beta \leq q+1$. Recall that any line of Σ meeting more than 2 of the β lines v_j , $1 \leq j \leq \beta$, meets all of them and, in particular, meets the lines v_1, v_2, v_3 . Since $\alpha \leq \beta \leq q+1 < |U|$ there exists a line u of U meeting at most 2 of the β lines v_j , $1 \leq j \leq \beta$. Since U and V have no lines in common, u is not also in V . Now U and V cover the same points. Thus, in order to cover the points of u , we must have $|V| \geq \beta + q - 1 \geq 2q - 1$, and we are done.

COMMENT. From the above, it would appear that rather more can perhaps be said from a structural point of view, especially in connection with the nets in [4]. We hope to pursue this further at a future date.

4. Translation nets. The work in the previous section is valid in $PG(3, q)$ where q is any prime power. In this section it is assumed that q is a prime, actually an odd prime. (For a brief discussion concerning the reason for making this assumption on q we refer to §6, the appendix.) To resume then, we are working in $\Sigma = PG(3, p)$ with p an odd prime. Equivalently we can think of $W = V_4(p)$, the underlying 4-dimensional vector space over $GF(p)$. It is then well known that any translation plane

of order p^2 is representable as a spread of Σ (or of W) and conversely (see [2]). We can apply Theorem 3 to obtain the following result.

THEOREM 4. *Let N be a net of order $n=p^2$ with p an odd prime. Assume that the deficiency of N is less than $2n^{1/2}-2$. Then N is embeddable in at most 2 translation planes π_1, π_2 . If π_1, π_2 exist, then π_1 is obtained by deriving π_2 , and vice versa.*

OUTLINE OF PROOF. We refer to [2] for a more detailed analysis. Let π_1 be a translation plane containing N . Choose a point O to be the origin of π_1 . Since $p \geq 3$, N contains 3 parallel classes which we may use to assign (Hall) coordinates to the points of π_1 , using O as the origin of coordinates. Since π_1 is a translation plane, we can then establish a bijection between the points of π_1 and the vectors of $W=V_4(p)$ and in this manner represent π_1 as a spread of W (or of $\Sigma=PG(3, p)$). Let π_2 be a different translation plane containing N . Let l be a line of π_2 through O such that l is not a line of N . We have already assigned a unique vector of W to each point of π_2 . Since π_2 is also a translation plane, the points of l yield an additive subset in W . This additive subset must actually be a subspace of W because p is a prime. Denote by U, V those lines of π_1, π_2 respectively which pass through O and are not lines of N . We can simultaneously think of U, V as partial spreads in W (and in Σ). Throwing out any lines common to the 2 partial spreads, we arrive at a stage where we can apply Theorem 3. Since we are assuming that the deficiency of N is less than $2n^{1/2}-2$ we conclude that U is a regulus, V is its opposite regulus and, therefore (see [2]), π_2 is obtained by deriving π_1 .

5. Combinatorial applications. Let $A=(a_{ij})$ be a $(0, 1)$ matrix of size $m \times n$, that is, A is an $m \times n$ matrix all of whose entries are either zero or one. We say that A is a *regulus matrix* if A contains no 4×4 submatrix having exactly 15 ones. More precisely, given any set $\{x_1, x_2, x_3, x_4\}$ of 4 distinct (but not necessarily consecutive) integers with $1 \leq x_1 < x_2 < x_3 < x_4 \leq m$ and another such set $\{y_1, y_2, y_3, y_4\}$, $1 \leq y_1 < y_2 < y_3 < y_4 \leq n$, then, of the sixteen matrix entries a_{x_i, y_j} ($1 \leq i, j \leq 4$) either all are ones or at least two are zeros. Note the connection between this definition and Theorem 1, whence the term regulus matrix. An argument similar to that of Theorem 3 then establishes the following result.

THEOREM 5. *Let A denote a regulus matrix of size $m \times n$. Let k denote some given positive integer. Assume that each row and each column of A contains exactly k ones. Then the following hold.*

- (1) $m=n=t \geq k$. If $t=k$, then all entries in A are 1.
- (2) If $t \neq k$ then $t \geq \max(k+1, 2k-4)$. Moreover for $k=q+1$, with q an odd prime power, this bound is sharp.

COMMENT. The examples of Foulser (see part (4) of Theorem 3) with $k=q+1$ and q an odd prime power show that the bound is sharp. However, other examples (to show the bound is sharp) are constructible with $k \neq q+1$, and hopefully, will be developed elsewhere. Also concerning the notion of a regulus matrix, we mention a paper of H. J. Ryser [6] where (0, 1) matrices which do not contain certain kinds of configurations are discussed.

6. **Appendix.** We want to make some brief concluding remarks, as follows.

A. Concerning the restriction on q in §4, an analysis of the proof of Theorem 4 shows that a slightly more general result is available. Namely, one can drop the assumption that $q=p$ providing one assumes that π_1 and π_2 , of order q^2 each have $GF(q)$ included in its kernel. The argument then goes through in this more general case, mutatis mutandis.

B. In the proof of the main result (Theorem 3, part (3)) we did not discuss the first case, namely the case when any 3 lines of U have at most 3 transversals in V (and, at the same time, each 3 lines of V have at most 3 transversals in U). We argue here as follows. For each 3-element subset E of the lines of U , we can count the number $n(E)$ of lines of V which are transversals to E . By hypothesis, $n(E) \leq 3$. Thus $\sum_E n(E) \leq 3 \binom{t}{3}$, the summation being over the $\binom{t}{3}$ subsets E (recall that $t=|U|=|V|$). On the other hand, each line of V meets exactly $q+1$ lines of U , so that $\sum_E n(E) = \binom{q+1}{3} t$. Hence $\binom{q+1}{3} t \leq 3 \binom{t}{3}$. From this, by straightforward computation, we obtain the desired result, namely that $t \geq 2q-2$.

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