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NOTE ON QUASIFIBRATIONS AND MANIFOLDS

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ABSTRACT. Let E be a closed C^∞-manifold which is the total space of a quasifibration over S^n with fibre S^k. Then, in many cases, E has the homotopy type of an orthogonal S^k-bundle over S^n. The proof includes a classification theorem for certain quasifibrations which has further applications.

1. **Introduction.** In [3], examples are given of the following phenomena: There exist quasifibrations over S^n with fibre S^k whose total spaces have the homotopy type of closed C^∞-manifolds but not the homotopy type of any S^k-bundle over S^n. In fact, for $k=2$ and $4<n\not\equiv 0 \bmod 4$, the complex $S^n \vee S^2 \cup e^{n+2}$ with the top cell attached by $[\iota^n, \iota^2]+\iota^2 \circ \alpha$ has the desired properties provided that $\alpha \in \pi_{n+1}(S^2)$, $\alpha\neq 0$, and some iterated suspension $E^N\alpha=0$. It quasifibers over S^n with fibre S^2 (by Theorem III of [8]), it has the homotopy type of a closed C^∞-manifold (by Browder-Novikov theory, for Poincaré duality holds and the Spivak normal spherical fibration is trivial since $E^N\alpha=0$) and it does not have the homotopy type of any S^2-bundle over S^n (for this would imply $\alpha=0$; see [3] for details). Thus S^k-quasifibrations over S^n lead to new examples of closed C^∞-manifolds, at least if $k=2$ and for some values of n. One may ask whether or not the same phenomena occurs for other values of k and n. The answer is "no" in the stable range.

THEOREM 1. *Let $p:E\to S^n$ be a quasifibration with fibre $\simeq S^k$ and $k\geqq n$; then E has the homotopy type of a closed C^∞-manifold iff it has the homotopy type of an orthogonal S^k-bundle over S^n.*

This follows from Theorem 6 below and the fact that if $k\geqq n$ there exists a homotopy section $s:S^n\to E$. We only consider quasifibrations which admit homotopy sections (as the examples above do) and whose total spaces have the homotopy type of a CW-complex. The main step is to classify them, as follows:

Two maps $p:E\to B$ and $p':E'\to B$ are called *equivalent*, $p\sim p'$, if there exists a homotopy equivalence $h:E\to E'$ such that $p'h\simeq p$; observe that

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this implies fibre homotopy equivalence if p and p' are fibrations. Given $\alpha \in \pi_{n+k-1}(S^k)$, n, $k \geqq 2$, let $X_\alpha = S^n \vee S^k \cup e^{n+k}$ with the top cell attached by $[\iota^n, \iota^k] + \iota^k \circ \alpha$, and let $p_\alpha : X_\alpha \to S^n$ be a map such that $p_\alpha | S^n \vee S^k$ is homotopic to the projection $S^n \vee S^k \to S^n$. The equivalence class of p_α is uniquely determined by α. Let $\bar{\alpha}$ be the coset of α modulo the subgroup which consists of all Whitehead products $[\rho, 1] \in \pi_{n+k-1}(S^k)$ with $\rho \in \pi_n(S^k)$.

THEOREM 2. (a) *The equivalence class of p_α contains a quasifibration $p: E \to S^n$ with $p^{-1}(*) = S^k$ which admits a homotopy section.* (b) *Let $\alpha, \beta \in \pi_{n+k-1}(S^k)$; then $p_\alpha \sim p_\beta$ iff $\bar{\alpha} = \bar{\beta}$ or $\bar{\alpha} = (-(-1) \circ \beta)^{-}$.*

COROLLARY. $\pi_q(X_\alpha) = \iota^n \circ \pi_q(S^n) + \iota^k \circ \pi_q(S^k)$ *if $q > 0$, and composition with ι^n and ι^q is injective.*

If $n > k+1$, (a) is contained in Theorem III of [**8**]; the general case is proved in §2 below. (b) follows from the definition of equivalence by a standard calculation which is left to the reader (recall $n, k \geqq 2$).

Let $p: E \to S^n$ be a quasifibration with fibre $\simeq S^k$ which admits a homotopy section. Let $C(p) \subset \pi_{n+k-1}(S^k)$ be the set of all "brace products" of p (compare [**4**] and §2 below).

THEOREM 3. *If $\alpha \in C(p)$, then $C(p)$ is the union of the cosets $\bar{\alpha}$ and $(-(-1) \circ \alpha)^{-}$, and $p \sim p_\alpha$.*

This completes the classification of quasifibrations over S^n with fibre $\simeq S^k$ which admit homotopy sections up to equivalence: The set of equivalence classes is in 1-1 correspondence with the factor set of $\pi_{n+k-1}(S^k)$ obtained by identifying $\alpha, \beta \in \pi_{n+k-1}(S^k)$ if $\bar{\alpha} = \bar{\beta}$ or $\bar{\alpha} = (-(-1) \circ \beta)^{-}$. The classification by homotopy type is as follows:

THEOREM 4. *Let $n \neq k$ or let $n = k$ be even; then $X_\alpha \simeq X_\beta$ iff $\bar{\alpha} = \pm \bar{\beta}$ or $\bar{\alpha} = (\pm(-1) \circ \beta)^{-}$. Let $n = k$ be odd; then $X_\alpha \simeq X_\beta$ iff there exists an integer m relatively prime to the order of $\bar{\alpha}$ such that $m\bar{\alpha} = \bar{\beta}$.*

The case $n \neq k$ is similar to the proof of (b) of Theorem 2 and is left to the reader; the case $n = k$ is discussed in full detail in [**6**].

Now let us return to the problem mentioned at the beginning. A quasifibration $p: E \to S^n$ is called *smoothable* if E has the homotopy type of a closed C^∞-manifold M; it is called *sectionally smoothable* if M and a homotopy equivalence $h: E \to M$ exist such that, for some homotopy section $s: S^n \to E$, $hs: S^n \to M$ is homotopic to a smooth embedding.

THEOREM 5. *A quasifibration over S^n with fibre $\simeq S^k$ which admits a homotopy section is sectionally smoothable iff it is equivalent to an orthogonal bundle.*

THEOREM 6. *Let $n \leqq 2k-3$; then a quasifibration over S^n with fibre $\simeq S^k$ which admits a homotopy section is smoothable iff it is equivalent to an orthogonal bundle.*

Theorem 6 follows from Theorem 5 and Haefliger's embedding theorem [**2**]. Theorems 2, 3 and 5 are proved in §2. Some further applications are given in §3.

2. **Proofs.** We further assume n, $k \geqq 2$. Let $u, q: S^{n-1} \times S^k \to S^k$ be maps such that q is the projection $q(x, y)=y$ and $u|S^{n-1} \vee S^k = q|S^{n-1} \vee S^k$. Let E_u be obtained from S^k by attaching $D^n \times S^k$ by u. Let $\chi: (D^n, S^{n-1}) \to (S^n, *)$ and $\phi: D^n \times S^k \to E_u$ be identification maps. Let $p_u: E_u \to S^n$ be $p_u(S^k)=*$ and $p_u\phi(x, y)=\chi(x)$ if $x \in D^n$, $y \in S^k$. By Theorem (1.8) of [**5**], p_u is a quasifibration with fibre S^k. The cross-section $S^n \to E_u$, given by $x \mapsto \phi(\chi^{-1}(x), *)$, and the inclusion $S^k \subset E_u$ define an imbedding $S^n \vee S^k \subset E_u$ with complement a $(n+k)$-cell. Thus $E_u = S^n \vee S^k \cup e^{n+k}$ and $p_u|S^n \vee S^k$ equals the projection $S^n \vee S^k \to S^n$. We are going to calculate the attaching map of e^{n+k}.

Let $\omega \in \pi_n(D^n, S^{n-1})$ be the class of the identity map, and let

$$\omega \times 1 \in \pi_{n+k}(D^n \times S^k, W) \quad \text{with } W = D^n \times * \cup S^{n-1} \times S^k$$

be the cross product of ω and $1 \in \pi_k(S^k)$, as defined in §5 of [**1**]. Define a diagram of maps

$$\begin{array}{ccccc}
W & \xrightarrow{\varphi} & S^{n-1} \times S^k/S^{n-1} \times * & \xleftarrow{\psi} & S^{n-1} \times S^k \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle \bar{u}} & \nwarrow{\scriptstyle h'} \ \searrow{\scriptstyle h} & \\
S^n \vee S^k & \xrightarrow{q'} & S^k & \xleftarrow{(1,f)} & S^k \vee S^{n+k-1}
\end{array}$$

as follows. ψ is the identification map, φ its extension to W such that $\varphi(D^n \times *)=*$. $\bar{u}$ is induced by u, q' is the restriction of q, and u' is the extension of u such that $u'(x, *)=\chi(x)$ if $x \in D^n$. h' is a map such that $h'(y)=\psi(*, y)$ if $y \in S^k$, and $h'|S^{n+k-1}$ represents the element $\varphi_*\partial(\omega \times 1)$. By an easy homology argument, h' is a homotopy equivalence; define h to be a homotopy inverse such that $h\psi(*, y)=y$ if $y \in S^k$. Then $h_*\varphi_*\partial(\omega \times 1)=\iota^{n+k-1}$, the class of the inclusion $S^{n+k-1} \to S^k \vee S^{n+k-1}$.

The class of the attaching map of the cell $e^{n+k} \subset E_u$ is $u'_*\partial(\omega \times 1)$; from (5.8) of [**1**] one easily gets

$$u'_*\partial(\omega \times 1) = [\iota^n, \iota^k] + \iota^k \circ q'_*u'_*\partial(\omega \times 1).$$

Now suppose u is defined by the equation $u=(1, f)h\psi$, where $f: S^{n+k-1} \to S^k$

represents a given element $\alpha \in \pi_{n+k-1}(S^k)$. Then

$$q'_* u'_* \partial(\omega \times 1) = \bar{u}_* \varphi_* \partial(\omega \times 1) = (1, f)_* h_* \varphi_* \partial(\omega \times 1)$$
$$= f_*(\iota^{n+k-1}) = \alpha;$$

thus the class of the attaching map of e^{n+k} is $[\iota^n, \iota^k] + \iota^k \circ \alpha$. This shows that the quasifibration p_u belongs to the equivalence class of p_α of §1 and proves (a) of Theorem 2.

We next prove Theorem 3. Let $p: E \to S^n$ be a quasifibration with fibre $\simeq S^k$ which admits a homotopy section. If $i: F = p^{-1}(*) \to E$ denotes the inclusion, there is a split exact sequence ($q > 0$)

$$0 \longrightarrow \pi_q(F) \xrightarrow{i_*} \pi_q(E) \xrightarrow{p_*} \pi_q(S^n) \longrightarrow 0.$$

Define $C(p) \subset \pi_{n+k-1}(S^k)$ as follows: Choose a homotopy section $s: S^n \to E$ and a homotopy equivalence $b: S^k \to F$. Denote by $\sigma \in \pi_n(E)$ and $\beta \in \pi_k(F)$ the homotopy class of s and b, respectively. Since the Whitehead product $[\sigma, i_*(\beta)] \in \pi_{n+k-1}(E)$ is mapped to 0 under p_*, the element

$$c(p; s, b) = -b_*^{-1} i_*^{-1}[\sigma, i_*(\beta)] \in \pi_{n+k-1}(S^k)$$

is well defined. Let $C(p)$ be the set of all $c(p; s, b)$, with s and b varying. If $b': S^k \to F$ is a homotopy equivalence, $b' \simeq b$ or $b' \simeq bd$, where $d: S^k \to S^k$ represents -1. In the first case $c(p; s, b') = c(p; s, b)$, in the second

$$c(p; s, b') = -d_*^{-1} b_*^{-1} i_*^{-1}[\sigma, -i_*(\beta)] = -(-1) \circ c(p; s, b).$$

If $s': S^n \to E$ is a homotopy section too, its homotopy class has the form $\sigma + i_* b_*(\rho)$ for some $\rho \in \pi_n(S^k)$, by the exact sequence above. Conversely, given ρ, each map representing $\sigma + i_* b_*(\rho)$ is a homotopy section. Now clearly the calculation

$$c(p; s', b) = -b_*^{-1} i_*^{-1}[\sigma + i_* b_*(\rho), i_*(\beta)] = c(p; s, b) - [\rho, 1]$$

proves the first part of Theorem 3.

Let $\alpha = c(p; s, b) \in C(p)$, and let $f: S^n \vee S^k \to E$ be the map which is s on S^n and b on S^k. Since

$$f_*([\iota^n, \iota^k] + \iota^k \circ \alpha) = [\sigma, i_*(\beta)] + i_* b_*(\alpha) = 0,$$

f extends to a map $X_\alpha \to E$ (recall the definition of X_α in §1). By the exact sequence above and the corollary of Theorem 2, any extension $X_\alpha \to E$ of f is a weak homotopy equivalence, hence a homotopy equivalence since we assume E to have the homotopy type of a CW-complex. By standard obstruction theory, an extension $g: X_\alpha \to E$ of f exists such that $pg \simeq p_\alpha$. Hence $p \sim p_\alpha$, in the sense of §1, which completes the proof of Theorem 3.

LEMMA 1. *Let $p: E \to S^n$ be an orthogonal S^k-bundle over S^n which admits a cross-section, and suppose $\delta \in \pi_{n-1}(SO(k))$ maps to a characteristic* *element of p in $\pi_{n-1}(SO(k+1))$. Then $J(\delta) \in C(p)$, with $J:\pi_{n-1}(SO(k)) \to \pi_{n+k-1}(S^k)$ the Hopf-Whitehead homomorphism.*

PROOF. Let $f:S^{n-1} \to SO(k+1)$ with $f(*)=1$ represent the image of δ, and let $u:S^{n-1} \times S^k \to S^k$ be the adjoint map $u(x, y)=f(x)(y)$. Then $u=q$ on $S^{n-1} \vee S^k$, and therefore $p_u:E_u \to S^n$ is defined and is an orthogonal S^k-bundle over S^n isomorphic to $p:E \to S^n$ (compare §3 of [**6**]). As is well known, $\bar{f}=u\psi^{-1}h'|S^{n+k-1}$ represents $J(\delta)$. Then $(1, \bar{f})h\psi \simeq u$ rel $S^{n-1} \times *$ and this yields $p \sim p_u \sim p_{J(\delta)}$, by calculating $u'_*\partial(\omega \times 1)$ as in the proof of Theorem 2. Now the lemma follows from Theorems 2 and 3.

REMARK. Since $((-1) \circ J(\delta))^{-}=(-J(\delta))^{-}$, by (2.2) of [**6**], the coset $C(p)=(J(\delta))^{-}$ is (up to an inclusion) just the $\lambda(p)$ of [**6**].

PROOF OF THEOREM 5. Let $\alpha \in \pi_{n+k-1}(S^k)$, $n, k \geqq 2$, and suppose there exists a closed C^∞-manifold M and a homotopy equivalence $h:X_\alpha \to M$ such that the element $h_*(\iota^n) \in \pi_n(M)$ is represented by a smoothly imbedded sphere $S^n \subset M$.

LEMMA 2. *Then there exists an orthogonal S^k-bundle $p:E \to S^n$ such that $p \sim p_\alpha$.*

This of course implies Theorem 5.

PROOF OF LEMMA 2. Let ν be the normal disc bundle of S^n in M with characteristic element $\delta \in \pi_{n-1}(SO(k))$. Its total space may be identified with the closure $\bar{U}$ of an open tubular neighborhood U of S^n in M. Let $q:M \to M/(M-U)$ be the projection onto the Thom space $M/(M-U)$ of ν. By Lemma 1 of [7] there exists a homotopy equivalence $s:M/(M-U) \to S^k \cup e^{n+k}$, where the cell e^{n+k} is attached by $J(\delta)$. Thus we get a map $f=sqh:X_\alpha \to S^k \cup e^{n+k}$ which we may assume to be cellular. Let $g:S^n \vee S^k \to S^k$ be the map defined by f. Since $f_*:H_{n+k}(X_\alpha) \to H_{n+k}(S^k \cup e^{n+k})$ is an isomorphism

$$g_*([\iota^n, \iota^k] + \iota^k \circ \alpha) = \varepsilon J(\delta) \quad \text{with } \varepsilon = \pm 1.$$

Furthermore, as will be shown below, $g_*(\iota^k)=\varepsilon'=\pm 1$. Hence

$$g_*([\iota^n, \iota^k] + \iota^k \circ \alpha) = [\varepsilon' g_*(\iota^n), 1] + (\varepsilon') \circ \alpha,$$

which together with the first equation implies $(J(\varepsilon\varepsilon'\delta))^{-}=(\varepsilon'(\varepsilon') \circ \alpha)^{-}$. Thus if $p:E \to S^n$ is the orthogonal S^k-bundle over S^n whose characteristic element is the image of $\varepsilon\varepsilon'\delta$, we get $p \sim p_\alpha$ by Lemma 1 and Theorems 2 and 3.

Let $\mathscr{H}$ be the natural homomorphism from homotopy to homology groups. To prove $g_*(\iota^k)=\pm 1$ it is sufficient to show that $f_*\mathscr{H}(\iota^k)$ generates $H_k(Y)$. This, since s is a homotopy equivalence and by excision, is equivalent to the statement that $j_*h_*\mathscr{H}(\iota^k)$ generates $H_k(M, M-U)$ with $j:(M, \varnothing) \to (M, M-U)$ the inclusion. Let $w \in H_{n+k}(X_\alpha)$ be a generator, and let $v \in H^n(X_\alpha)$ be dual to $\mathscr{H}(\iota^n) \in H_n(X_\alpha)$; thus the Kronecker product $(\mathscr{H}(\iota^n), v)=1$. Then, as follows from the cell decomposition of X_α, the cap product $v \cap w = \pm \mathscr{H}(\iota^k)$. Hence $y \cap z = \pm h_* \mathscr{H}(\iota^k)$, if $z \in H_{n+k}(M)$ is a generator and if $y \in H^n(M)$ is dual to $h_* \mathscr{H}(\iota^n) \in H_n(M)$. Therefore we have to prove that $j_*(y \cap z)$ generates $H_k(M, M-U)$. By excision, this is equivalent to the following: If z' generates $H_{n+k}(\bar{U}, \partial U)$ and if $i: \bar{U} \to M$ is the inclusion, then $i^*(y) \cap z'$ generates $H_k(\bar{U}, \partial U)$. This is true by Lefschetz duality, since $i^*(y)$ generates $H^n(\bar{U})$.

3. **Further applications.** First, from the cell decomposition of X_α and the Corollary of Theorem 2, we get the following examples, generalizing Theorem (2.10) of [**3**]:

THEOREM 7. *Let $\alpha \in \pi_{n+k-1}(S^k)$, $n, k \geqq 2$. If $n=k$ is even, suppose the Hopf invariant $H(\alpha)=0$. Then X_α and $S^n \times S^k$ have isomorphic homotopy groups and integral cohomology rings, but are of different homotopy type if $\bar{\alpha} \neq 0$.*

The next theorems concern the connection between homotopy type and fibre homotopy type. Instead of Theorems 2 and 3 one may use [**9**] (and Theorem 4) to prove them; but this is too difficult a way to get the results below, since it includes a deep theorem of G. W. Whitehead (Theorem (3.2) of [**10**], as corrected in [**11**]). Observe that our proofs use only elementary homotopy constructions.

The difficult point in Theorems 2 and 3 is the calculation of the automorphism $\alpha \mapsto (-1) \circ \alpha$ of $\pi_{n+k-1}(S^k)$. In the special case $k=2$, $n \geqq 2$, this automorphism is the identity, which implies

THEOREM 8. *Fibrations which admit cross-sections, with fibre a 2-sphere and with base any sphere, are fibre homotopically equivalent, if their total spaces are of the same homotopy type.*

This fails to be true if cross-sections do not exist.

The fibrations of Theorem 8 are interesting for the following reason. Let $p: E \to S^n$ be a S^2-fibration with cross-section which is not trivial (in the sense of fibre homotopy type). Then p is not locally trivial if $n \geqq 3$. For otherwise its structure group could be reduced from $\mathrm{Top}(S^2)$ to $O(2)$ contradicting $\pi_{n-1}(O(2))=0$ if $n \geqq 3$ (compare the proof of Theorem (2.6) of [**3**]). The lowest dimensional example of such a fibration, whose fibre homotopy type does not contain any locally trivial fibration, occurs if $n=3$: It is p_α with α the generator of $\pi_4(S^2)=Z_2$. Its total space X_α is a 1-connected 5-dimensional Poincaré duality space which does not have the homotopy type of a closed C^∞-manifold (compare [**3**]; this does not follow from our results).

Let $p:E \to S^n$ be a fibration with fibre $\simeq S^k$ which admits a cross-section. Let p_m be induced from p by a map $S^n \to S^n$ of degree m. If $\alpha \in C(p)$, then $m\alpha \in C(p_m)$, as easily follows from the definition of $C(p)$. Hence we get, from Theorems 2, 3, 4,

THEOREM 9. *Let $p:E \to S^n$, $p':E' \to S^n$ be fibrations which admit cross-sections with fibre $\simeq S^k$. Then, if $E \simeq E'$, p' is induced from p (up to fibre homotopy equivalence) by some map $S^n \to S^n$.*

COROLLARIES. (a) *If E has the homotopy type of an orthogonal bundle, then p has the fibre homotopy type of an orthogonal bundle.*

(b) *If $E \simeq S^n \times S^k$, then p is trivial (in the sense of fibre homotopy type).*

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