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INVERTIBLE MEASURE PRESERVING TRANSFORMATIONS AND POINTWISE CONVERGENCE

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ABSTRACT. An investigation of pointwise convergence of sequences $\{\sum_{j=-\infty}^{\infty} a_j^k f(T^{-j}x):k=1,2,\cdots\}$ where f lies in the space $L^1([0,1])$ of Lebesgue integrable functions on the unit interval, T is an invertible measure preserving transformation on $[0,1]$, and the sequence of polynomials $\{\sum_{j=-\infty}^{\infty} a_j^k z^{-j}:k=1,2,\cdots\}$ is uniformly bounded and pointwise convergent for all z such that $|z|=1$.

Spectral properties. An invertible measure preserving transformation T on the unit interval I is known to induce a unitary operator on the space $L^2(I)$ of square integrable functions on I [**6**, p. 13]. By the spectral theorem [**5**, p. 71] there exists a spectral measure E on the Borel subsets of the unit circle C in the complex plane such that for any integer k, $U^k=\int z^k E(dz)$ in the sense of strong convergence. Let the resolution of the identity E_t, t in $[0,2\pi)$, be given by $E(\{\exp(is):0\leqq s<t\})$. Then [**3**, p. 482]

$$E_t=\sum_{j\neq 0}\frac{\exp(ijt)-1}{2\pi ij}U^{-j}+\frac{t}{2\pi}+\frac{E(\{1\})-E(\{\exp(it)\})}{2}$$

where, for each z in C, $E(\{z\})=\lim(\sum_{j=-n}^{n} z^j U^{-j})/(2n+1)$ and the symbol $\sum_{j\neq 0}$ denotes the limit as n tends to infinity of the sum $\sum_{j=-n;j\neq 0}^{n}$.

Substituting the Fourier series

$$\begin{aligned}\pi-\sum_{j\neq 0}\frac{\exp(ijs)}{ij}&=s, &&0<s<2\pi,\\ &=\pi, &&s=0,\end{aligned}$$

on the right-hand side of the identity [**1**, p. 100]

$$s=\pi+32\sum_{j=0}^{\infty}\frac{\sin(\frac{1}{4}(2j+1)s)-(-1)^j\cos(\frac{1}{4}(2j+1)s)}{\pi^2(2j+1)^3}\qquad(0\leqq s\leqq 2\pi),$$

and then integrating both sides with respect to the spectral measure

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for the unitary operator $\exp(-it)U$ yields

$$E_t = \frac{t}{2\pi} + \tfrac{1}{2}E(\{1\}) - \sum_{j\neq 0} \frac{U^{-j}}{2\pi ij}$$
$$- \frac{16}{\pi^3}\sum_{j=0}^{\infty} i\,\frac{\exp(-i(2j+1)t/4)U^{(2j+1)/4} - \exp(i(2j+1)t/4)U^{-(2j+1)/4}}{(2j+1)^3}$$
$$- \frac{16}{\pi^3}\sum_{j=0}^{\infty} (-1)^j\{(\exp(-i(2j+1)t/4)U^{(2j+1)/4})/(2j+1)^3$$
$$+ (\exp(i(2j+1)t/4)U^{-(2j+1)/4})/(2j+1)^3\}.$$

By the uniform boundedness of the series [7, p. 18] we can justify taking the integral inside the summation signs above.

The unitary operators $U^{k/4}$, $k=0, \pm 1, \pm 2, \cdots$, are defined by $U^{k/4}=\int z^{k/4}E(dz)$. Thus the convolution property for the spectral measure of a unitary operator with the multiplicative property [**4**, pp. 639, 640] permits us to establish that, since U is multiplicative, then so is $U^{k/4}$. For if f, g and their product fg lie in $L^2(I)$ then

$$U^{k/4}fg = \int z^{k/4}E(dz)fg = \iint z^{k/4}E(w^{-1}\,dz)fE(dw)g$$
$$= \int \left(\int (w^{-1}z)^{k/4}E(w^{-1}\,dz)f\right) w^{k/4}E(dw)g = (U^{k/4}f)(U^{k/4}g).$$

Hence if f lies in $L^2(I)$ with L^1 norm $\|f\|_1$ then there exists g in $L^2(I)$ with L^2 norm $\|g\|_2$ such that $f=g^2$, $\|f\|_1=\|g\|_2^2$, and $\|U^{k/4}f\|_1=\|U^{k/4}g\|_2^2=\|f\|_1$. Using the identity above for E_t it now follows that there exists a constant K such that for any collection $\{B_m: m=1, 2, \cdots\}$ of disjoint half-open interval subsets of C and any f in $L^2(I)$ we have $\|E(\bigcup B_m)f\|_1 \leqq K\|f\|_1$. By the usual measure theoretic argument (Dinculeanu [2]), for any f in $L^2(I)$ and any Borel subset B of C, $\|E(B)f\|_1 \leqq K\|f\|_1$. Since $L^2(I)$ is dense in $L^1(I)$ we extend by continuity the operator E to $L^1(I)$ and so (retaining the symbol E for the extension) $\|E(B)f\|_1 \leqq K\|f\|_1$ for all f in $L^1(I)$ and Borel subset B of C. Note that the space $L^\infty(I)$ of essentially bounded functions on I lies in $L^2(I)$. Hence E is defined on $L^\infty(I)$. We now deduce that for any h in $L^\infty(I)$ with L^∞ norm $\|h\|_\infty$ and any Borel B in C, $\|E(B)h\|_\infty \leqq K\|h\|_\infty$. For if f lies in $L^1(I)$, using (f, h) to denote the integral of the product $f\bar{h}$ (where $\bar{h}$ is the complex conjugate of h) over I, we get $(E(B)f, h)=(f, E(B)h)$ which is clear if f lies in $L^2(I)$ and extends to $L^1(I)$ by continuity.

Next let us show the existence of a constant K' such that for any h in $L^\infty(I)$ and any sequence $\{B_k\}$ of disjoint Borel subsets of C, $\|\sum |E(B_k)h|\|_\infty \leqq K'\|h\|_\infty$. Otherwise there would exist some finite family

$\{B_k : k=1, 2, \cdots, n\}$ of disjoint Borel subsets of C such that for some h in $L^2(I)$, $\sum |E(B_k)h|$ is "much" greater than $\|h\|_\infty$ on some subset X of I of positive measure. Hence by considering the real and imaginary parts of $E(B_k)h$ and all possible subsequences of $\{B_k : k=1, 2, \cdots, n\}$, we see that there must exist some subsequence $\{B_{k_j}\}$ for which either the real or imaginary part of $E(\bigcup B_{k_j})h$ is "much" greater in absolute value than $\|h\|_\infty$ on a subset of X of positive measure, i.e. $\|E(\bigcup B_{k_j})h\|_\infty > K\|h\|_\infty$ which is a contradiction.

By now we have that for any given h in $L^\infty(I)$, $E(\cdot)h(x)$ is a complex measure on the Borel subsets of C with total variation not exceeding $K'\|h\|_\infty$ [**8**, p. 117] for almost all x in I. Hence we can define in the usual way the integral $\int q(x, z)E(dz)h(x)$ of a bounded Borel measurable function $q(x, z)$ on $I \times C$ to yield an essentially bounded function of x, i.e. an element of $L^\infty(I)$. Furthermore if $\{q_k(x, z)\}$ is a pointwise convergent uniformly bounded sequence of Borel measurable functions then by Lebesgue's dominated convergence theorem the integrals $\int q_k(x, z)E(dz)h(x)$ form a uniformly bounded (in $L^\infty(I)$) almost everywhere pointwise convergent sequence of functions on I.

Convergence properties. Consider a sequence of polynomials $p_k(z) = \sum_{j=-\infty}^{\infty} a_j^k z^{-j}$, $k=1, 2, \cdots$, where z lies in C and a_j^k are complex coefficients all but a finite number of which vanish. For a given function f in $L^1(I)$ define $p_k(U)f$ to be $\sum_{j=-\infty}^{\infty} U^{-j}(a_j^k f)$, i.e. $\sum_{j=-\infty}^{\infty} a_j^k U^{-j} f$.

THEOREM. *If U is an operator on $L^1(I)$ induced by an invertible measure preserving transformation on the unit interval I and $\{p_k(z) : k=1, 2, \cdots\}$ a pointwise convergent sequence of uniformly bounded (trigonometric) polynomials on the unit circle then, for all f in $L^1(I)$, $p_k(U)f(x)$ converges pointwise for almost all x in I as k tends to infinity.*

PROOF. If $p_k(U)f$ does not converge pointwise almost everywhere, there exists a nonzero constant d such that for all x in a subset Y of I of positive measure $|Y|$

$$\sup_{k', k'' \geqq m} |p_{k'}(U)f(x) - p_{k''}(U)f(x)| > d$$

for all integers m. Hence given any m there exists an integer $M > m$ and measurable functions $k'(x)$, $k''(x)$; $m \leqq k'(x), k''(x) \leqq M$ such that for some function h, $|h| = 1$, we have

$$\left(\sum_j U^{-j}(a_j^{k'(x)} - a_j^{k''(x)})f(x), h(x)\right) > \frac{d\,|Y|}{2}.$$

Note that M was chosen to make

$$\sup_{m\leq k',k''\leq M} |p_{k'}(U)f(x) - p_{k''}(U)f(x)| > d$$

for all x in a subset of Y of measure greater than $|Y|/2$. But by the measure preserving property of the operator inducing U we have

$$\left(\sum_j U^{-j}((a_j^{k'(x)} - a_j^{k''(x)})f(x)), h(x)\right) = \left(f(x), \sum_j (a_j^{k'(x)} - a_j^{k''(x)})U^j h(x)\right)$$
$$= \left(f(x), \int \sum (a_j^{k'(x)} - a_j^{k''(x)}) z^j E(dz) h(x)\right)$$

and by the discussion at the end of the previous section this tends to zero as m tends to infinity, which is a contradiction. Q.E.D.

The above could be generalized to not necessarily invertible transformations on the real line, which would make Birkhoff's ergodic theorem [**6**, p. 18] a special case of the theorem above by taking the polynomial $\sum_{j=-k}^{k} z^{-j}/(2k+1)$ for $p_k(z)$. In fact we could go even further by considering operators which are L^1 and L^2 contractions with the multiplicative property by using the generalized spectral measures associated with them [**9**, pp. 12–18].

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