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ON A CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. We show that the class $\mathfrak{E}_0$ of analytic functions f in a plane region $\Omega \notin O_{AB}$ vanishing at $z_0 \in \Omega$ and such that $1/f$ omits a set of values of area $\geqq \pi$ is not compact. Here O_{AB} denotes the class of Riemann surfaces which have no nonconstant bounded analytic functions. We remark that the extremal functions maximizing $|f'(z_0)|$ in $\mathfrak{E}_0$ coincide with linear transformations $w/(1-cw)$ of those for the same problem in the class $\mathfrak{B}_0$ consisting of functions f such that $f(z_0)=0$ and $|f(z)|\leqq 1$, i.e. so-called Ahlfors functions. Here $1/c$ is an omitted value of the Ahlfors function.

Under the notations in the above abstract Ahlfors and Beurling **[1]** stated that the classes $\mathfrak{B}_0$ and $\mathfrak{E}_0$ are both compact and proved that the maxima of $|f'(z_0)|$ in $\mathfrak{B}_0$ and $\mathfrak{E}_0$ are equal. However, we can show that the alleged compactness of $\mathfrak{E}_0$ is not true by constructing a counterexample: For the annulus $\frac{1}{2}<|z|<2$, the functions $f_n=(\frac{3}{4})^n(z^n-(\frac{3}{2})^n)/z^n$, $n=1, 2, \cdots$, belong to $\mathfrak{E}_0$ with $z_0=\frac{3}{2}$. Then $\{f_n\}$ tends to zero for $\frac{9}{8}+\delta<|z|<2$ and to infinity for $\frac{1}{2}<|z|<\frac{9}{8}-\delta$, $\delta>0$ as $n\to\infty$.

If $\Omega \notin O_{AB}$, there exist the extremal functions $A(z)$ which maximize $|f'(z_0)|$ in $\mathfrak{B}_0$. Those functions are called the *Ahlfors functions* which are unique save for rotations [3]. If $1/c$ is an omitted value of $A(z)$, $A(z)/(1-cA(z))$ belongs to $\mathfrak{E}_0$. By the result cited above it is extremal for the problem in $\mathfrak{E}_0$.

For any extremal $g \in \mathfrak{E}_0$, let E be the set of all omitted values of g. From the extremality of g the area of E is equal to π. They used a transformation

$$\Phi\left(\frac{1}{g}\right) = \frac{1}{\pi}\iint_E \frac{du\,dv}{\frac{1}{g}-w}, \qquad w = u + iv,$$

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and proved $\Phi(1/g) \in \mathfrak{B}_0$ with $\Phi'(1/g(z_0)) = g'(z_0)$ [**1**]. Hence $\Phi(1/g)$ is an Ahlfors function and there exists a point $\omega \in E$ such that $|\Phi(\omega)| = 1$. Note that $\Phi(w)$ is continuous in the whole plane [**2**]. Without loss of generality (by a rotation) we set $\Phi(\omega) = 1$ and $E_+ = E \cap \{\mathrm{Re}(w - \omega) \geqq 0\}$. Then from the equality statement for Schwarz's inequality in their proof we infer that E_+ coincides with the disc $r \leqq 2 \cos \theta$, $|\theta| \leqq \pi/2$ $(w - \omega = re^{i\theta})$, except for a set of area zero. We can deduce, from $\Phi(\omega) = 1$, that the area of $E - E_+$ vanishes. Denoting by c the center of the above disc, by a direct calculation we see that $\Phi(w)$ reduces to a linear transformation $1/(w-c)$ for $|w-c| \geqq 1$. Hence we have $g = \Phi(1/g)/(1 + c\Phi(1/g))$. Clearly $-1/c$ is an omitted value of the Ahlfors function $\Phi(1/g)$ and therefore g is of the form stated in the abstract.

References

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