PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, March 1974

GROWTH AND DECAY OF SOLUTIONS OF $y^{(2n)} - py = 0$

T. T. READ

ABSTRACT. Simple estimates of the rate of growth and decay of certain solutions of $y^{(2n)}-py=0$ on $[0, \infty)$ when p is eventually nonnegative are used to obtain sufficient conditions for the existence of exponential solutions, solutions which approach 0, and $L^2(0, \infty)$ solutions.

We shall give an elementary estimate of the rate of growth of certain solutions of

$$y^{(2n)} - py = 0 \tag{1}$$

when p is an eventually nonnegative continuous function on $[0, \infty)$, and an estimate of the rate of decay of solutions y of (1) such that for some x_0,

$$(-1)^j y^{(j)}(x) \geqq 0 \quad \text{for } j = 0, 1, \cdots, 2n-1 \text{ and all } x \geqq x_0. \tag{2}$$

It is a result of Hartman and Wintner [1] that there is a solution satisfying (2).

These estimates yield immediately a generalization, in the sharpest possible form, of a result of C. R. Putnam [6] on the existence of exponentially increasing and decreasing solutions of (1) when p is eventually bounded away from 0 (Theorem 3). The estimate for the rate of decay of solutions of the form (2) is then applied to establish a sufficient condition for (1) to have a solution in $L^2(0, \infty)$ (Theorem 5), and a necessary and sufficient condition for a solution satisfying (2) to approach 0 (Theorem 4). For $n=1$, Theorem 4 is due to Hille [4].

One common method of proving the existence of $L^2(0, \infty)$ solutions is to verify that for some $C>0$, Ly $(=y^{(2n)}-py$ here) satisfies $\|Ly\| \geqq C\|y\|$ for all y with compact support in $(0, \infty)$. ($\|\cdot\|$ denotes the L^2 norm.) In this case (1) has at least n linearly independent L^2 solutions (see for example Naĭmark [5]). From the constant coefficient equation $y^{(4)}-y=0$, to which both Theorems 4 and 5 apply, it is clear that we cannot hope to obtain this many L^2 solutions or indeed to show, under the respective hypotheses of Theorems 4 and 5, that every bounded solution of (1)

Received by the editors February 26, 1973 and, in revised form, July 11, 1973.
AMS (MOS) subject classifications (1970). Primary 34C10.
Key words and phrases. Exponential growth, monotonic solution, $L^2(0, \infty)$ solution.

approaches 0 or is in $L^2(0, \infty)$. Nevertheless, the existence of even a single L^2 solution can be of considerable physical interest.

We begin by stating as a lemma the version of the result of Hartman and Wintner [1] that we need.

LEMMA. *Let p be nonnegative and continuous on* $[0, \infty)$. *Then*

$$y^{(2n)} - py = 0, \qquad y(0) = 1 \tag{3}$$

has a solution z such that $(-1)^j z^{(j)} \geqq 0$ on $[0, \infty)$ for $j=0, 1, \cdots, 2n$.

When $n=1$ it is clear that z is unique and is the only bounded solution of (3), since any solution y such that $y'(0)>0$ is increasing and unbounded.

Our basic estimate is now easily established. When $n=1$ it is essentially Theorem 9.2.1 of [3].

THEOREM 1. *Let p and q be distinct continuous functions on $[0, \infty)$ such that $p \geqq q \geqq 0$. If y_p and y_q are positive solutions of* (3) *and*

$$y^{(2n)} - qy = 0, \qquad y(0) = 1 \tag{4}$$

respectively, and if $y_p^{(j)}(0) \geqq y_q^{(j)}(0)$ for $j=1, 2, \cdots, 2n-1$, then $y_p^{(j)} \geqq y_q^{(j)}$ for $j=0, 1, \cdots, 2n$ and $y_p - y_q \to \infty$ as $x \to \infty$.

PROOF. Suppose first that $y_p'(0) > y_q'(0)$. Set $g = y_p - y_q$. Then $g(0)=0$, $g'(0)>0$, $g^{(j)}(0) \geqq 0$ for $j=2, \cdots, 2n-1$, and

$$\begin{aligned} g^{(2n)} &= y_p^{(2n)} - y_q^{(2n)} = py_p - qy_q \\ &\geqq q(y_p - y_q) = qg. \end{aligned}$$

Since $g'(0)>0$, g is positive on some interval $(0, \varepsilon)$. But then $g(x)>0$ for all x, since otherwise $g^{(2n)}$ must change sign before g does. It follows that $g^{(2n)} \geqq 0$ and hence that $g^{(j)} \geqq 0$ for $j=1, 2, \cdots, 2n-1$. Finally, $g(x) \to \infty$ as $x \to \infty$ since g' is positive and nondecreasing.

If $y_p'(0) = y_q'(0)$, then for $m=1, 2, \cdots$ let $y_{p,m}$ be the solution of (3) such that $y_{p,m}'(0) = y_p'(0) + 1/m$, $y_{p,m}^{(j)}(0) = y_p^{(j)}(0)$ for $j=2, 3, \cdots, 2n-1$. Then for any x, $y_p(x) = \lim y_{p,m}(x) \geqq y_q(x)$. Hence $g = y_p - y_q \geqq 0$ and, as before, each $g^{(j)} \geqq 0$. Since $p \neq q$, we must have $g' \neq 0$ and so $g \to \infty$. This completes the proof.

COROLLARY 1. *If $p \geqq q \geqq 0$ and if y_p is any solution of* (3) *such that $y_p^{(j)}(0) > 0$ for $j=1, 2, \cdots, 2n-1$, then every solution y of* (4) *satisfies $|y| \leqq Ky_p$.*

PROOF. It is clear that a set of $2n$ linearly independent solutions y of (4) can be found each of which satisfies $0 < y^{(j)}(0) \leqq y_p^{(j)}(0)$ for $j=1, 2, \cdots, 2n-1$.

COROLLARY 2. *If* $p \geqq q \geqq 0$, *if* $n=1$, *and if* z_p *and* z_q *are the unique solutions of* (3) *and* (4) *respectively which satisfy* (2), *then* $z_p(x)<z_q(x)$ *for* $x>0$.

PROOF. If $z_p'(0) \geqq z_q'(0)$, then $z_p - z_q \to \infty$. Hence $z_p < z_q$ on some interval $(0, \varepsilon)$. If for some $x_0>0$, $z_p(x_0)=z_q(x_0)$ while $z_p(x)<z_q(x)$ for $x \leqq x_0$, then again $z_p - z_q \to \infty$. Hence $z_p(x)<z_q(x)$ for $x>0$.

Theorem 1 can be put in the following more suggestive form. For convenience we shall, for any sufficiently differentiable function h, write h_j for the polynomial in the derivatives of h given by $h_j=e^{-h}D^j(e^h)$. Note that $h_0=1$.

THEOREM 2. *Let h be a C^{2n} function on* $[0, \infty)$ *such that* $h_{2n} \geqq 0$ *and* $h_j>0$ *for* $j=1, 2, \cdots, 2n-1$. *Suppose* $p \geqq h_{2n}$. *Then*

(a) $y^{(j)} \geqq Kh_j e^h$ *for* $j=0, 1, \cdots, 2n$ *whenever y is a solution of* (3) *such that* $y^{(j)}(0) \geqq h_j(0)$ *for* $j=1, 2, \cdots, 2n-1$.

(b) $0 \leqq (-1)^j z^{(j)} \leqq L h_{2n-j-1}^{-1} e^{-h}$ *for* $j=0, 1, \cdots, 2n-1$ *whenever z is a solution of* (3) *which satisfies* (2).

PROOF. We may assume $h(0)=0$. Then (a) is simply a restatement of Theorem 1 with $q=h_{2n}$ and $y_q=e^h$. Let z be a solution of (3) which satisfies (2) and let y be a solution of (1) such that $y^{(j)}(0) \geqq h_j(0)$ for $j=1, 2, \cdots, 2n-1$. The function $\sum_{j=0}^{2n-1} (-1)^j z^{(j)} y^{(2n-j-1)}$ is constant since its derivative is zero, and each term of the sum is nonnegative. Thus for each j, $0 \leqq (-1)^j z^{(j)} \leqq C/y^{(2n-j-1)}$ and (b) follows from (a).

In particular we have

COROLLARY 3. *If, in addition to the hypotheses of Theorem* 2, h_{2n-1} *is bounded away from* 0, *then there are solutions y and z of* (3) *such that* $y \geqq Ke^h$, $0 \leqq z \leqq Le^{-h}$.

In this notation Corollary 1 becomes

COROLLARY 4. *Let h be as in Theorem* 2. *Suppose* $0 \leqq p \leqq h_{2n}$. *Then every solution y of* (3) *satisfies* $|y^{(j)}| \leqq Kh_j e^h$ *for* $j=0, 1, \cdots, 2n$.

The hypotheses on h in all the above results are satisfied, for all sufficiently large x, whenever h is a polynomial whose term of highest degree has a positive coefficient. The special case $h(x)=rx$, $r>0$ yields

THEOREM 3. *If* $\liminf p(x)>r^{2n}$, *then there are solutions y and z of* (1) *and* $x_0 \geqq 0$ *such that for all* $x \geqq x_0$, $y(x) \geqq Ke^{rx}$, *and* $0 \leqq z(x) \leqq Le^{-rx}$.

The case $n=1$ is due to C. R. Putnam [**6**], although it is not clear that his proof gives the connection between $\liminf p$ and the exponent.

We will now suppose that p is eventually nonnegative and use Theorem 2 to investigate the behavior of a solution of (1) which satisfies (2).

THEOREM 4. *Let p be eventually nonnegative. If z is a solution of* (1) *which satisfies* (2), *then* $z(x)$ *approaches* 0 *as* $x\to\infty$ *if and only if* $\int_0^\infty t^{2n-1}p(t)\,dt=\infty$.

PROOF. We may assume that $p(x)\geqq 0$ for all x. Suppose first that $\int_0^\infty t^{2n-1}p(t)\,dt=\infty$. Let $f(x)=\int_0^x t^{2n-1}p(t)\,dt$, and let v be the function such that $v^{(2n-1)}=f$, $v(0)=1$, and $v^{(j)}(0)=0$ for $j=1, 2, \cdots, 2n-1$. Now set $h=\log v$. Then each h_j is positive and $h_{2n}(x)=x^{2n-1}p(x)/v(x)$. Since $f(x)\to\infty$ as $x\to\infty$, $v(x)\geqq x^{2n-1}$ for all large x. Then $p(x)\geqq h_{2n}(x)$ and by Theorem 2 we have eventually $0\leqq z\leqq Lh_{2n-1}^{-1}e^{-h}=Kf^{-1}\to 0$.

Now suppose that $\int_0^\infty t^{2n-1}p(t)\,dt<\infty$. It is a theorem of Haupt [**2**] that for any solution y of (1), $y^{(2n-1)}(x)$ approaches a finite limit as $x\to\infty$. Choose $c>0$ so that the solution y_c with $y_c^{(j)}(0)=c$ for $j=0, 1, \cdots, 2n-1$ satisfies $y_c^{(2n-1)}\to\frac{1}{2}$. Then for all sufficiently large x,

$$y_c^{(j)}(x) \leqq x^{2n-j-1}/(2n-j-1)!, \qquad j = 0, 1, \cdots, 2n-1.$$

Let z be a solution of (1) satisfying (2). Then $\sum_{j=0}^{2n-1}(-1)^j z^{(j)} y_c^{(2n-j-1)}=M$ is constant. If $z(x)\to 0$, then using the above estimate for the $y_c^{(j)}$ yields that for all large x,

$$\sum_{j=1}^{2n-1}(-1)^j z^{(j)}(x)x^{j-1}/j! \geqq M/2x.$$

Since each term on the left is positive, $\int_0^\infty(-1)^j z^{(j)}(x)x^{j-1}\,dx=\infty$ for some j. Now an integration by parts yields that

$$\int_0^\infty(-1)^{j-1}z^{(j-1)}(x)x^{j-2}\,dx = \infty.$$

By induction, $\int_0^\infty -z'(x)\,dx=\infty$. But this contradicts the boundedness of z. Hence z cannot approach 0 and the proof is complete.

When $n=1$ there is a unique bounded solution. Thus we have the following result of E. Hille [**4**].

COROLLARY 5. *Let p be eventually nonnegative. The equation* $y''-py=0$ *has a solution which approaches* 0 *as* $x\to\infty$ *if and only if* $\int_0^\infty tp(t)\,dt=\infty$.

In a somewhat similar fashion we can establish

THEOREM 5. *Let p be eventually nonnegative. If z is a solution of* (1) *which satisfies* (2), *and if for some* $c>[(3)(5)\cdots(4n-1)/2^{2n-2}]^{1/2}$ *and all large* x, $\int_0^x [t^{2n-1}p(t)]^{1/2}\,dt\geqq c\sqrt{x}$, *then* $z\in L^2(0,\infty)$.

PROOF. Note that if $d=2n-\frac{1}{2}$, then

$$d(d-1)\cdots(d-2n+2)/(2n-d) = (3)(5)\cdots(4n-1)/2^{2n-2}.$$

Choose $d \in (2n-\frac{1}{2}, 2n)$ so that $c^2 > d(d-1)\cdots(d-2n+2)/(2n-d)$. Then by the Schwarz lemma we have for large x that

$$c^2 x \leqq \left[\int_0^x [t^{2n-1}p(t)]^{1/2}\,dt\right]^2$$
$$\leqq \int_0^x t^{2n-1-d}\,dt \int_0^x t^d p(t)\,dt = \frac{1}{2n-d}\,x^{2n-d}\int_0^x t^d p(t)\,dt.$$

Hence

$$\int_0^x t^d p(t)\,dt \geqq (2n-d)c^2 x^{d-2n+1} > d(d-1)\cdots(d-2n+2)x^{d-2n+1}. \tag{5}$$

Now set $f(x)=\int_0^x t^d p(t)\,dt$ and $h=\log v$, where $v^{(2n-1)}=f$, $v(0)=1$, and $v^{(j)}(0)=0$ for $j=1, 2, \cdots, 2n-1$. Then, as in the proof of Theorem 5, $h_{2n}(x)=x^d p(x)/v(x)$. By (5), $h_{2n}(x)\leqq p(x)$ for all large x. Hence we have eventually that $0\leqq z\leqq K/h_{2n-1}^{-1}e^{-h}=Kf^{-1}$. Since $d-2n+1>\frac{1}{2}$, $f^{-1}\in L^2(0, \infty)$ and the proof is complete.

One situation in which the hypothesis of Theorem 5 is satisfied is

Corollary 6. *If* $\liminf x^{2n}p(x)>a^2>2^{-2n}(3)(5)\cdots(4n-1)$, *then a solution z of* (1) *which satisfies* (2) *is in* $L^2(0, \infty)$.

Proof. For large t, $t^{2n}p(t)>a^2$ or $[t^{2n-1}p(t)]^{1/2}>at^{-1/2}$. Hence for sufficiently large x, $\int_0^x [t^{2n-1}p(t)]^{1/2}\,dt\geqq 2a\sqrt{x}$ and Theorem 5 may be applied.

The function $(1+x)^{-1/2}$ satisfies (1) with

$$p(x) = (3)(5)\cdots(4n-1)/2^{2n}(x+1)^{2n}.$$

Thus the hypotheses of Theorem 5 and Corollary 6 cannot be weakened even to the extent of allowing equality. Moreover, when $n=1$ we have the following partial converse of Corollary 6.

Corollary 7. *Let p be eventually nonnegative. If* $\limsup x^2p(x)<\frac{3}{4}$, *then no solution z of $y''-py=0$ is in* $L^2(0, \infty)$.

Proof. We may assume that $0\leqq p(x)<\frac{3}{4}(x+1)^2$ for all x. The unique bounded solution z of $y''-py=0$, $y(0)=1$ satisfies (2). By Corollary 2, $z(x)\geqq(x+1)^{-1/2}$. Thus the equation has no $L^2(0, \infty)$ solutions.

References

1. P. Hartman and A. Wintner, *Linear differential and difference equations with monotone solutions*, Amer. J. Math. 75 (1953), 731–743. MR **15,** 221.

2. O. Haupt, *Über das asymptotische Verhalten der Lösungen gewisser linearer gewöhnlicher Differentialgleichungen*, Math. Z. **48** (1942), 289–292. MR **4,** 276.

3. E. Hille, *Lectures on ordinary differential equations*, Addison-Wesley, Reading, Mass., 1969. MR **40** #2939.

4. ———, *Non-oscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), 234–252. MR **10,** 376.

5. M. A. Naĭmark, *Linear differential operators.* Part II: *Linear differential operators in Hilbert space*, GITTL, Moscow, 1954; English transl., Ungar, New York, 1968. MR **16,** 702; **41** #7485.

6. C. R. Putnam, *On isolated eigenfunctions associated with bounded potentials*, Amer. J. Math. **72** (1950), 135–147. MR **11,** 517.

Department of Mathematics, Western Washington State College, Bellingham, Washington 98225