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GROWTH AND DECAY OF SOLUTIONS OF $y^{(2n)} - py = 0$

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ABSTRACT. Simple estimates of the rate of growth and decay of certain solutions of $y^{(2n)}-py=0$ on $[0, \infty)$ when p is eventually nonnegative are used to obtain sufficient conditions for the existence of exponential solutions, solutions which approach 0, and $L^2(0, \infty)$ solutions.

We shall give an elementary estimate of the rate of growth of certain solutions of

$$y^{(2n)} - py = 0 \tag{1}$$

when p is an eventually nonnegative continuous function on $[0, \infty)$, and an estimate of the rate of decay of solutions y of (1) such that for some x_0,

$$(-1)^j y^{(j)}(x) \geqq 0 \quad \text{for } j = 0, 1, \cdots, 2n-1 \text{ and all } x \geqq x_0. \tag{2}$$

It is a result of Hartman and Wintner [1] that there is a solution satisfying (2).

These estimates yield immediately a generalization, in the sharpest possible form, of a result of C. R. Putnam [6] on the existence of exponentially increasing and decreasing solutions of (1) when p is eventually bounded away from 0 (Theorem 3). The estimate for the rate of decay of solutions of the form (2) is then applied to establish a sufficient condition for (1) to have a solution in $L^2(0, \infty)$ (Theorem 5), and a necessary and sufficient condition for a solution satisfying (2) to approach 0 (Theorem 4). For $n=1$, Theorem 4 is due to Hille [4].

One common method of proving the existence of $L^2(0, \infty)$ solutions is to verify that for some $C>0$, Ly $(=y^{(2n)}-py$ here) satisfies $\|Ly\|\geqq C\|y\|$ for all y with compact support in $(0, \infty)$. ($\|\cdot\|$ denotes the L^2 norm.) In this case (1) has at least n linearly independent L^2 solutions (see for example Naĭmark [5]). From the constant coefficient equation $y^{(4)}-y=0$, to which both Theorems 4 and 5 apply, it is clear that we cannot hope to obtain this many L^2 solutions or indeed to show, under the respective hypotheses of Theorems 4 and 5, that every bounded solution of (1)

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approaches 0 or is in $L^2(0, \infty)$. Nevertheless, the existence of even a single L^2 solution can be of considerable physical interest.

We begin by stating as a lemma the version of the result of Hartman and Wintner [1] that we need.

LEMMA. *Let p be nonnegative and continuous on* $[0, \infty)$. *Then*

$$y^{(2n)} - py = 0, \qquad y(0) = 1 \tag{3}$$

has a solution z such that $(-1)^j z^{(j)} \geqq 0$ on $[0, \infty)$ for $j=0, 1, \cdots, 2n$.

When $n=1$ it is clear that z is unique and is the only bounded solution of (3), since any solution y such that $y'(0)>0$ is increasing and unbounded.

Our basic estimate is now easily established. When $n=1$ it is essentially Theorem 9.2.1 of [3].

THEOREM 1. *Let p and q be distinct continuous functions on* $[0, \infty)$ *such that $p \geqq q \geqq 0$. If y_p and y_q are positive solutions of* (3) *and*

$$y^{(2n)} - qy = 0, \qquad y(0) = 1 \tag{4}$$

respectively, and if $y_p^{(j)}(0) \geqq y_q^{(j)}(0)$ for $j=1, 2, \cdots, 2n-1$, then $y_p^{(j)} \geqq y_q^{(j)}$ for $j=0, 1, \cdots, 2n$ and $y_p - y_q \to \infty$ as $x \to \infty$.

PROOF. Suppose first that $y_p'(0) > y_q'(0)$. Set $g = y_p - y_q$. Then $g(0)=0$, $g'(0)>0$, $g^{(j)}(0) \geqq 0$ for $j=2, \cdots, 2n-1$, and

$$\begin{aligned} g^{(2n)} &= y_p^{(2n)} - y_q^{(2n)} = py_p - qy_q \\ &\geqq q(y_p - y_q) = qg. \end{aligned}$$

Since $g'(0)>0$, g is positive on some interval $(0, \varepsilon)$. But then $g(x)>0$ for all x, since otherwise $g^{(2n)}$ must change sign before g does. It follows that $g^{(2n)} \geqq 0$ and hence that $g^{(j)} \geqq 0$ for $j=1, 2, \cdots, 2n-1$. Finally, $g(x) \to \infty$ as $x \to \infty$ since g' is positive and nondecreasing.

If $y_p'(0) = y_q'(0)$, then for $m=1, 2, \cdots$ let $y_{p,m}$ be the solution of (3) such that $y_{p,m}'(0) = y_p'(0) + 1/m$, $y_{p,m}^{(j)}(0) = y_p^{(j)}(0)$ for $j=2, 3, \cdots, 2n-1$. Then for any x, $y_p(x) = \lim y_{p,m}(x) \geqq y_q(x)$. Hence $g = y_p - y_q \geqq 0$ and, as before, each $g^{(j)} \geqq 0$. Since $p \neq q$, we must have $g' \neq 0$ and so $g \to \infty$. This completes the proof.

COROLLARY 1. *If $p \geqq q \geqq 0$ and if y_p is any solution of* (3) *such that $y_p^{(j)}(0) > 0$ for $j=1, 2, \cdots, 2n-1$, then every solution y of* (4) *satisfies $|y| \leqq Ky_p$.*

PROOF. It is clear that a set of $2n$ linearly independent solutions y of (4) can be found each of which satisfies $0 < y^{(j)}(0) \leqq y_p^{(j)}(0)$ for $j=1, 2, \cdots, 2n-1$.

COROLLARY 2. *If* $p \geqq q \geqq 0$, *if* $n=1$, *and if* z_p *and* z_q *are the unique solutions of* (3) *and* (4) *respectively which satisfy* (2), *then* $z_p(x) < z_q(x)$ *for* $x > 0$.

PROOF. If $z_p'(0) \geqq z_q'(0)$, then $z_p - z_q \to \infty$. Hence $z_p < z_q$ on some interval $(0, \varepsilon)$. If for some $x_0 > 0$, $z_p(x_0) = z_q(x_0)$ while $z_p(x) < z_q(x)$ for $x \leqq x_0$, then again $z_p - z_q \to \infty$. Hence $z_p(x) < z_q(x)$ for $x > 0$.

Theorem 1 can be put in the following more suggestive form. For convenience we shall, for any sufficiently differentiable function h, write h_j for the polynomial in the derivatives of h given by $h_j = e^{-h} D^j(e^h)$. Note that $h_0 = 1$.

THEOREM 2. *Let* h *be a* C^{2n} *function on* $[0, \infty)$ *such that* $h_{2n} \geqq 0$ *and* $h_j > 0$ *for* $j = 1, 2, \cdots, 2n-1$. *Suppose* $p \geqq h_{2n}$. *Then*

(a) $y^{(j)} \geqq K h_j e^h$ *for* $j = 0, 1, \cdots, 2n$ *whenever* y *is a solution of* (3) *such that* $y^{(j)}(0) \geqq h_j(0)$ *for* $j = 1, 2, \cdots, 2n-1$.

(b) $0 \leqq (-1)^j z^{(j)} \leqq L h_{2n-j-1}^{-1} e^{-h}$ *for* $j = 0, 1, \cdots, 2n-1$ *whenever* z *is a solution of* (3) *which satisfies* (2).

PROOF. We may assume $h(0) = 0$. Then (a) is simply a restatement of Theorem 1 with $q = h_{2n}$ and $y_q = e^h$. Let z be a solution of (3) which satisfies (2) and let y be a solution of (1) such that $y^{(j)}(0) \geqq h_j(0)$ for $j = 1, 2, \cdots, 2n-1$. The function $\sum_{j=0}^{2n-1} (-1)^j z^{(j)} y^{(2n-j-1)}$ is constant since its derivative is zero, and each term of the sum is nonnegative. Thus for each j, $0 \leqq (-1)^j z^{(j)} \leqq C / y^{(2n-j-1)}$ and (b) follows from (a).

In particular we have

COROLLARY 3. *If, in addition to the hypotheses of Theorem* 2, h_{2n-1} *is bounded away from* 0, *then there are solutions* y *and* z *of* (3) *such that* $y \geqq K e^h$, $0 \leqq z \leqq L e^{-h}$.

In this notation Corollary 1 becomes

COROLLARY 4. *Let* h *be as in Theorem* 2. *Suppose* $0 \leqq p \leqq h_{2n}$. *Then every solution* y *of* (3) *satisfies* $|y^{(j)}| \leqq K h_j e^h$ *for* $j = 0, 1, \cdots, 2n$.

The hypotheses on h in all the above results are satisfied, for all sufficiently large x, whenever h is a polynomial whose term of highest degree has a positive coefficient. The special case $h(x) = rx$, $r > 0$ yields

THEOREM 3. *If* $\liminf p(x) > r^{2n}$, *then there are solutions* y *and* z *of* (1) *and* $x_0 \geqq 0$ *such that for all* $x \geqq x_0$, $y(x) \geqq K e^{rx}$, *and* $0 \leqq z(x) \leqq L e^{-rx}$.

The case $n = 1$ is due to C. R. Putnam [**6**], although it is not clear that his proof gives the connection between $\liminf p$ and the exponent.

We will now suppose that p is eventually nonnegative and use Theorem 2 to investigate the behavior of a solution of (1) which satisfies (2).

THEOREM 4. *Let p be eventually nonnegative. If z is a solution of* (1) *which satisfies* (2), *then $z(x)$ approaches* 0 *as $x \to \infty$ if and only if* $\int_0^\infty t^{2n-1}p(t)\,dt=\infty$.

PROOF. We may assume that $p(x) \geqq 0$ for all x. Suppose first that $\int_0^\infty t^{2n-1}p(t)\,dt=\infty$. Let $f(x)=\int_0^x t^{2n-1}p(t)\,dt$, and let v be the function such that $v^{(2n-1)}=f$, $v(0)=1$, and $v^{(j)}(0)=0$ for $j=1, 2, \cdots, 2n-1$. Now set $h=\log v$. Then each h_j is positive and $h_{2n}(x)=x^{2n-1}p(x)/v(x)$. Since $f(x) \to \infty$ as $x \to \infty$, $v(x) \geqq x^{2n-1}$ for all large x. Then $p(x) \geqq h_{2n}(x)$ and by Theorem 2 we have eventually $0 \leqq z \leqq L h_{2n-1}^{-1} e^{-h} = Kf^{-1} \to 0$.

Now suppose that $\int_0^\infty t^{2n-1}p(t)\,dt<\infty$. It is a theorem of Haupt [2] that for any solution y of (1), $y^{(2n-1)}(x)$ approaches a finite limit as $x \to \infty$. Choose $c>0$ so that the solution y_c with $y_c^{(j)}(0)=c$ for $j=0, 1, \cdots, 2n-1$ satisfies $y_c^{(2n-1)} \to \frac{1}{2}$. Then for all sufficiently large x,

$$y_c^{(j)}(x) \leqq x^{2n-j-1}/(2n-j-1)!, \qquad j = 0, 1, \cdots, 2n-1.$$

Let z be a solution of (1) satisfying (2). Then $\sum_{j=0}^{2n-1} (-1)^j z^{(j)} y_c^{(2n-j-1)} = M$ is constant. If $z(x) \to 0$, then using the above estimate for the $y_c^{(j)}$ yields that for all large x,

$$\sum_{j=1}^{2n-1} (-1)^j z^{(j)}(x) x^{j-1}/j! \geqq M/2x.$$

Since each term on the left is positive, $\int_0^\infty (-1)^j z^{(j)}(x) x^{j-1}\,dx=\infty$ for some j. Now an integration by parts yields that

$$\int_0^\infty (-1)^{j-1} z^{(j-1)}(x) x^{j-2}\,dx = \infty.$$

By induction, $\int_0^\infty -z'(x)\,dx=\infty$. But this contradicts the boundedness of z. Hence z cannot approach 0 and the proof is complete.

When $n=1$ there is a unique bounded solution. Thus we have the following result of E. Hille [4].

COROLLARY 5. *Let p be eventually nonnegative. The equation $y''-py=0$ has a solution which approaches* 0 *as $x \to \infty$ if and only if* $\int_0^\infty tp(t)\,dt=\infty$.

In a somewhat similar fashion we can establish

THEOREM 5. *Let p be eventually nonnegative. If z is a solution of* (1) *which satisfies* (2), *and if for some* $c>[(3)(5)\cdots(4n-1)/2^{2n-2}]^{1/2}$ *and all large x,* $\int_0^x [t^{2n-1}p(t)]^{1/2}\,dt \geqq c\sqrt{x}$, *then* $z \in L^2(0, \infty)$.

PROOF. Note that if $d=2n-\frac{1}{2}$, then

$$d(d-1)\cdots(d-2n+2)/(2n-d) = (3)(5)\cdots(4n-1)/2^{2n-2}.$$

Choose $d\in(2n-\frac{1}{2}, 2n)$ so that $c^2>d(d-1)\cdots(d-2n+2)/(2n-d)$. Then by the Schwarz lemma we have for large x that

$$c^2x \leqq \left[\int_0^x [t^{2n-1}p(t)]^{1/2}\,dt\right]^2$$
$$\leqq \int_0^x t^{2n-1-d}\,dt\int_0^x t^dp(t)\,dt = \frac{1}{2n-d}x^{2n-d}\int_0^x t^dp(t)\,dt.$$

Hence

$$(5)\qquad \int_0^x {}^dp(t)\,dt \geqq (2n-d)c^2x^{d-2n+1} > d(d-1)\cdots(d-2n+2)x^{d-2n+1}.$$

Now set $f(x)=\int_0^x t^dp(t)\,dt$ and $h=\log v$, where $v^{(2n-1)}=f$, $v(0)=1$, and $v^{(j)}(0)=0$ for $j=1, 2, \cdots, 2n-1$. Then, as in the proof of Theorem 5, $h_{2n}(x)=x^dp(x)/v(x)$. By (5), $h_{2n}(x)\leqq p(x)$ for all large x. Hence we have eventually that $0\leqq z\leqq K/h_{2n-1}^{-1}e^{-h}=Kf^{-1}$. Since $d-2n+1>\frac{1}{2}$, $f^{-1}\in L^2(0,\infty)$ and the proof is complete.

One situation in which the hypothesis of Theorem 5 is satisfied is

COROLLARY 6. *If* $\liminf x^{2n}p(x)>a^2>2^{-2n}(3)(5)\cdots(4n-1)$, *then a solution* z *of* (1) *which satisfies* (2) *is in* $L^2(0,\infty)$.

PROOF. For large t, $t^{2n}p(t)>a^2$ or $[t^{2n-1}p(t)]^{1/2}>at^{-1/2}$. Hence for sufficiently large x, $\int_0^x [t^{2n-1}p(t)]^{1/2}\,dt\geqq 2a\sqrt{x}$ and Theorem 5 may be applied.

The function $(1+x)^{-1/2}$ satisfies (1) with

$$p(x) = (3)(5)\cdots(4n-1)/2^{2n}(x+1)^{2n}.$$

Thus the hypotheses of Theorem 5 and Corollary 6 cannot be weakened even to the extent of allowing equality. Moreover, when $n=1$ we have the following partial converse of Corollary 6.

COROLLARY 7. *Let* p *be eventually nonnegative. If* $\limsup x^2p(x)<\frac{3}{4}$, *then no solution* z *of* $y''-py=0$ *is in* $L^2(0,\infty)$.

PROOF. We may assume that $0\leqq p(x)<\frac{3}{4}(x+1)^2$ for all x. The unique bounded solution z of $y''-py=0$, $y(0)=1$ satisfies (2). By Corollary 2, $z(x)\geqq(x+1)^{-1/2}$. Thus the equation has no $L^2(0,\infty)$ solutions.

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