PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, March 1974

REGULAR FACTORIZATIONS OF CONTRACTIONS

BÉLA SZ.-NAGY AND CIPRIAN FOIAŞ

ABSTRACT. Equivalent conditions are given for the regularity of a factorization of a contraction, two of which exhibit immediately the duality property of this notion.

The concept of regular factorization of contractions of Hilbert spaces[1] was introduced in [1] by the authors in connection with their investigations on the invariant subspace problem; cf. [2, §VII.3].

Let A_0 be a contraction of a Hilbert space $\mathfrak{A}$ into a Hilbert space $\mathfrak{A}_*$, and let

$$A_0 = A_2 A_1 \tag{F}$$

be a factorization of A_0 as a product of a contraction A_1 of $\mathfrak{A}$ into some "intermediate" Hilbert space $\mathfrak{B}$, and of a contraction A_2 of $\mathfrak{B}$ into $\mathfrak{A}_*$:

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "A_0"] \arrow[dr, "A_1"'] & & \mathfrak{A}_* \\ & \mathfrak{B} \arrow[ur, "A_2"'] & \end{tikzcd}$$

Define the corresponding "defect operators" by

$$D_j = (I_j - A_j^* A_j)^{1/2}, \qquad D_{*j} = (I_{*j} - A_j A_j^*)^{1/2} \qquad (j = 0, 1, 2),$$

where I_j and I_{*j} denote the identity operators on the space of definition of A_j and A_j^*, respectively.

The factorization (F) was defined to be *regular* if condition (i) given in the Proposition below holds. Some basic arithmetical properties of such factorizations were established in [2], in particular it was proven that (F) is regular if and only if its dual

$$A^* = A_1^* A_2^* \tag{F$_*$}$$

Received by the editors December 15, 1972.
AMS (MOS) subject classifications (1970). Primary 47A50; Secondary 47A15.
Key words and phrases. Hilbert space, contraction, factorization, defect operators, dual factorization.

[1] The Hilbert spaces considered are either all real or all complex. Separability is not assumed. Contraction means linear operator of norm $\leqq 1$.

is regular. This property does not follow easily from condition (i). This makes it desirable to find conditions equivalent to (i) and exhibiting the duality property. We are going to give two such conditions (ii) and (iii); the equivalence of (i) and (iii) was already stated (without proof) by Švarcman [3].[2]

PROPOSITION. *For the factorization* (F) *the following three conditions are equivalent*:

(i) $$\overline{\{D_2A_1a \oplus D_1a : a \in \mathfrak{A}\}} = \overline{D_2\mathfrak{B}} \oplus \overline{D_1\mathfrak{A}},$$

(ii) $$\overline{\{D_2b \oplus D_{*1}b : b \in \mathfrak{B}\}} = \overline{D_2\mathfrak{B}} \oplus \overline{D_{*1}\mathfrak{B}},$$

(iii) $$D_2\mathfrak{B} \cap D_{*1}\mathfrak{B} = \{0\}.$$

The proof will be given in the steps (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i).

(1) We make use of the decomposition $(D_{*1}\mathfrak{B})^{-} = (A_1D_1\mathfrak{A})^{-} \oplus \ker A_1^*$ generated by the contraction A_1; cf. [**2**, formula I.3.7]. Observe that for any $b \in \ker A_1^*$ we have $b = D_{*1}b$, and hence

$$0 \oplus b = (D_2b \oplus D_{*1}b) - (D_2b \oplus 0).$$

As we obviously have $D_2b \oplus 0 \in (D_2\mathfrak{B})^{-} \oplus (A_1D_1\mathfrak{A})^{-}$, we conclude that condition (ii) holds if (and only if) every element of the form $D_2b \oplus A_1D_1a$ ($a \in \mathfrak{A}$, $b \in \mathfrak{B}$) is the limit of a sequence of elements of the form $D_2b_n \oplus D_{*1}b_n$ ($b_n \in \mathfrak{B}$). Now if condition (i) is satisfied then there exists a sequence $a_n \in \mathfrak{A}$ such that

$$D_2A_1a_n \to D_2b \quad \text{and} \quad D_1a_n \to D_1a.$$

Hence, setting $b_n = A_1a_n (\in \mathfrak{B})$ we have

$$\begin{aligned} D_2b_n \oplus D_{*1}b_n &= D_2A_1a_n \oplus D_{*1}A_1a_n = D_2A_1a_n \oplus A_1D_1a_n \\ &\to D_2b \oplus A_1D_1a. \end{aligned}$$

Therefore, (i) *implies* (ii).

(2) Let us notice that for any selfadjoint operator S on a Hilbert space $\mathfrak{H}$ we have $\mathfrak{H} = (S\mathfrak{H})^{-} \oplus \ker S$. Hence it follows that $S\mathfrak{H} = S((S\mathfrak{H})^{-})$. Thus we have in particular $D_2\mathfrak{B} = D_2((D_2\mathfrak{B})^{-})$ and $D_{*1}\mathfrak{B} = D_{*1}((D_{*1}\mathfrak{B})^{-})$. Hence, for every element $x \in D_2\mathfrak{B} \cap D_{*1}\mathfrak{B}$ there exist elements $b' \in (D_2\mathfrak{B})^{-}$ and $b'' \in (D_{*1}\mathfrak{B})^{-}$ such that $x = D_2b' = D_{*1}b''$. Set $y = b' \oplus (-b'')$. Clearly y belongs to $(D_2\mathfrak{B})^{-} \oplus (D_{*1}\mathfrak{B})^{-}$. Moreover, it is orthogonal to all elements

[2] ADDED IN PROOF (October 31, 1973). In the meantime the proof appeared in a paper of the same title in Mat. Issled. **8** (1973), 201–216, at p. 210. This proof of the implication (iii)$\Rightarrow$(i) is identical with ours. However, the equivalent form (ii) does not occur in Švarcman's paper.

of the form $D_2b \oplus D_{*1}b$ ($b \in \mathfrak{B}$), because

$$(y, D_2b \oplus D_{*1}b) = (b', D_2b) - (b'', D_{*1}b)$$
$$= (D_2b' - D_{*1}b'', b) = (x - x, b) = 0.$$

Thus if (ii) is satisfied we must have $y=0$, and hence $b'=0$, $b''=0$; consequently $x=0$.

Therefore, (ii) *implies* (iii).

(3) Let $v \oplus u$ be any element of $(D_2\mathfrak{B})^- \oplus (D_1\mathfrak{A})^-$, orthogonal to all elements of the form $D_2A_1a \oplus D_1a$ ($a \in \mathfrak{A}$). This means that

$$(\alpha) \qquad A_1^*D_2v + D_1u = 0;$$

hence we deduce

$$A_1A_1^*D_2v + A_1D_1u = 0, \qquad D_2v - D_{*1}^2D_2v + D_{*1}A_1u = 0$$

so that

$$(\beta) \qquad D_2v = D_{*1}(D_{*1}D_2v - A_1u).$$

If (iii) is satisfied then (β) implies $D_2v=0$; as $v \in (D_2\mathfrak{B})^-$ we conclude that $v=0$. Then (α) implies $D_1u=0$, and again since $u \in (D_1\mathfrak{A})^-$, we conclude that $u=0$. Thus $v \oplus u$ is zero.

Therefore, (iii) *implies* (i).

This finishes the proof.

REFERENCES

1. B. Sz.-Nagy and C. Foiaş, *Une caractérisation des sous-espaces invariants pour une contraction de l'espace de Hilbert*, C.R. Acad. Sci. Paris, **258** (1964), 3426–3429. MR **29** #1537.

2. ———, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris; Akad. Kiadó, Budapest, 1967; English rev. transl., North-Holland, Amsterdam; American Elsevier, New York; Akad. Kiadó, Budapest, 1970. MR **37** #778; **43** #947.

3. Ja. S. Švarcman, *On invariant subspaces of a dissipative operator and the divisors of its characteristic function*, Funkcional. Anal. i Priložen. **4** (1970), no. 4, 85–86= Functional Anal. Appl. **4** (1970), 342–343. MR **43** #3832.

BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, SZEGED, HUNGARY

MATHEMATICS INSTITUTE, ACADEMY OF SCIENCES, BUCHAREST, ROMANIA