

## A UNIFIED TREATMENT OF SOME THEOREMS ON POSITIVE MATRICES

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**ABSTRACT.** Various theorems on positive matrices are shown to be corollaries of one general theorem, the proof of which bears on Legendre functions, as used in Rockafellar's *Convex analysis*.

**1. Introduction: The main theorem.** Let  $X$  be a finite set,  $(\mu(x))_{x \in X}$  strictly positive numbers, and  $H$  a fixed linear subspace of  $\mathbf{R}^X$ . We shall prove the following:

**THEOREM 1.** *There exists a unique (nonlinear) map from  $\mathbf{R}^X$  to  $H$ , denoted  $f \mapsto h_f$ , such that*

$$\sum_{x \in X} [\exp f(x) - \exp h_f(x)] g(x) \mu(x) = 0$$

for all  $g$  in  $H$ .

**2. A first application: Matrices with prescribed marginals.** Given an  $n$ -sequence  $r = (r_i)_{i=1}^n$  and an  $m$ -sequence  $s = (s_j)_{j=1}^m$  of nonnegative numbers such that  $\sum_{i=1}^n r_i = \sum_{j=1}^m s_j$ , denote by  $\mathcal{M}(r, s)$  the set of  $(n, m)$  matrices  $(a_{ij})$  with  $a_{ij} \geq 0$  such that  $r_i = \sum_{j=1}^m a_{ij}$  and  $s_j = \sum_{i=1}^n a_{ij}$  for all  $i=1, 2, \dots, n$  and  $j=1, 2, \dots, m$ . Also, let  $E = \{1, 2, \dots, n\}$  and  $F = \{1, 2, \dots, m\}$ . We define the linear map  $c$  from  $\mathbf{R}^E \oplus \mathbf{R}^F$  to  $\mathbf{R}^{E \times F}$  by:

$$c[(b_i)_{i \in E}, (b'_j)_{j \in F}] = (b_i + b'_j)_{(i,j) \in E \times F}.$$

If  $X$  is a subset of  $E \times F$ ,  $\pi$  denotes the canonical map from  $\mathbf{R}^{E \times F}$  to  $\mathbf{R}^X$ , i.e.

$$\pi[(a_{ij})_{(i,j) \in E \times F}] = (a_{ij})_{(i,j) \in X}.$$

We say also that  $X$  is an  $(r, s)$  pattern if there exists  $(a_{ij})$  in  $\mathcal{M}(r, s)$  such that  $X = \{(i, j); a_{ij} > 0\}$ .

Now the first corollary to the Theorem 1 is:

**COROLLARY 1.** *Let two sequences  $r = (r_i)_{i=1}^n$  and  $s = (s_j)_{j=1}^m$  of nonnegative numbers with  $\sum_{i=1}^n r_i = \sum_{j=1}^m s_j$ ,  $M$  an  $(n, m)$  matrix with  $\mu_{ij} \geq 0$  and  $X = \{(i, j); \mu_{ij} > 0\}$ .*

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Then there exist two diagonal matrices

$$D_b = (e^{b_1}, e^{b_2}, \dots, e^{b_n})$$

and

$$D_{b'} = (e^{b'_1}, \dots, e^{b'_m})$$

with positive diagonal elements such that  $D_bMD_{b'}$  is in  $\mathcal{M}(r, s)$  if and only if  $X$  is an  $(r, s)$  pattern. Furthermore, the set  $(b, b')$  of  $\mathbf{R}^E \oplus \mathbf{R}^F$ , such that  $D_bMD_{b'}$  is in  $\mathcal{M}(r, s)$ , is exactly a translate of  $\text{Ker}[\pi \circ c]$  if nonempty.

PROOF. If  $b$  and  $b'$  exist, the fact that  $X$  is an  $(r, s)$  pattern is obvious. Conversely, suppose that  $X$  is an  $(r, s)$  pattern. Then there exists  $(a_{ij}) \in \mathcal{M}(r, s)$  with  $X = \{(i, j); a_{ij} > 0\}$ . Denote  $f_{ij} = \text{Log}(a_{ij}/\mu_{ij})$  when  $(i, j) \in X$  and apply the theorem to this  $f \in \mathbf{R}^X$  and to  $H$ , the range in  $\mathbf{R}^X$  of  $\pi \circ c$ . Then there exists  $h = (h_{ij})_{(i,j) \in X}$  in  $H$  such that

$$\sum_{(i,j) \in X} a_{ij} g_{ij} = \sum_{(i,j) \in X} \exp(h_{ij}) g_{ij} \mu_{ij}$$

for all  $(g_{ij})_{(i,j) \in X}$  of  $H$ , and such  $h$  is unique. Writing now  $h_{ij} = b_i + b'_j$  for some  $(b_i)_{i=1}^n \in \mathbf{R}^E$  and  $(b'_j)_{j=1}^m \in \mathbf{R}^F$ , we have  $(e^{b_i} \mu_{ij} e^{b'_j})_{(i,j) \in E} \in \mathcal{M}(r, s)$ .

All suitable  $(b, b')$  must satisfy  $\pi \circ c(b, b') = h$ , and this ends the proof.

In order to complete Corollary 1 we have to specify  $\text{Ker}[\pi \circ c]$  for a given  $X \subset E \times F$ . We index  $E$  and  $F$  such that  $E \cap F = \emptyset$ . We consider the linear graph (nonoriented) with  $E \cup F$  as the set of vertices and  $X$  as the set of edges. The connected components of that linear graph can be written  $(E_\alpha \cup F_\alpha)_{\alpha \in A}$ , where  $A$  is some index set (with  $E_\alpha \cup F_\alpha$  nonempty, but  $E_\alpha$  or  $F_\alpha = \emptyset$  can happen), and we call  $(E_\alpha)_{\alpha \in A}$  and  $(F_\alpha)_{\alpha \in A}$  the partitions of  $E$  and  $F$  associated to  $X$ .

PROPOSITION 1. Let  $X$  be a subset of  $E \times F$ ,  $\pi$  the canonical map from  $\mathbf{R}^{E \times F}$  to  $\mathbf{R}^X$ ,  $(E_\alpha)_{\alpha \in A}$  and  $(F_\alpha)_{\alpha \in A}$  the associated partitions of  $E$  and  $F$ . Then  $\dim[\text{Ker}(\pi \circ c)] = \text{number of } \alpha \text{ such that } E_\alpha \times F_\alpha \text{ is not empty}$ . Furthermore, when  $(i_1, j_1)$  and  $(i_2, j_2)$  are in  $E_\alpha \times F_\alpha$  and  $(b, b') \in \text{Ker}[\pi \circ c]$ , then

$$b_{i_1} = b_{i_2} = -b_{j_1} = -b_{j_2}.$$

PROOF. Suppose  $E_\alpha \times F_\alpha$  is nonempty; take  $(i_1, j_1)$  and  $(i_2, j_2)$  in  $E_\alpha \times F_\alpha$  and  $(b, b')$  in  $\text{Ker}(\pi \circ c)$ . From the definition of  $E_\alpha \cup F_\alpha$ , either there exists a sequence of  $X$

$$(e_0, f_0), (e_1, f_0), (e_1, f_1), (e_2, f_1), \dots, (e_n, f_{n-1})$$

or there exists a sequence of  $X$

$$(e_0, f_0), (e_0, f_1), (e_1, f_1), (e_1, f_2), \dots, (e_n, f_n)$$

with  $e_0 = c_1$  and  $e_n = c_2$ .

Since  $b_e + b'_f = 0$  when  $(e, f) \in X$  this implies that  $b_{i_1} = b_{i_2}$ . In the same way we prove  $b'_{j_1} = b'_{j_2}$ . To see that  $b_{i_1} = -b'_{j_1}$ , we find a path from  $i_1$  to  $j_1$  in an analogous manner. Then  $\dim \text{Ker}[\pi \circ c]$  is not larger than the number of nonempty  $E_\alpha \times F_\alpha$ . From this, it is easy to finish the proof.

REMARKS. Corollary 1 was conjectured in 1964 by P. Thionet who is a French statistician [10], and in a weaker form in 1960 [9]. Proofs in particular cases are given from [1]. Independently, the case  $n=m, r_i=s_j=1$  was intensively studied in a paper by R. Sinkhorn in 1964 [7]; see [3] for a recent proof and references.<sup>1</sup>

It is easy to prove the following:  $X$  is an  $(r, s)$  pattern if and only if for any  $x \in X$  there exists an  $(r, s)$  pattern  $Y(x)$  such that  $x \in Y(x) \subset X$ . An interesting feature of the case of doubly stochastic matrices is that the patterns  $Y(x)$  are associated in a natural way to permutation matrices, permitting a characterisation, via a König theorem, of a corresponding pattern as a direct sum of so-called fully indecomposable matrices [3]. It would be interesting to get such a combinatorial characterisation for a general  $(r, s)$  pattern.

**3. Second application: A theorem due to D. J. Hartfiel.** Let  $E = \{1, \dots, n\}$  and  $\mathcal{M}_s$  be the set of  $n$ -square matrices  $(a_{ij})$  with  $a_{ij} \geq 0$  and  $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$  for all  $i$  in  $E$ . We define the linear map  $d$  from  $\mathbf{R}^E$  to  $\mathbf{R}^{E \times E}$  by:

$$d[(b_i)_{i \in E}] = (b_i - b_j)_{(i,j) \in E \times E}.$$

If  $X$  is a subset of  $E \times E$ ,  $\pi$  is the canonical map from  $\mathbf{R}^{E \times E}$  to  $\mathbf{R}^X$ . We say also that  $X$  is a *symmetric* pattern if there exists  $(a_{ij})$  in  $\mathcal{M}_s$  such that  $X = \{(i, j); a_{ij} > 0\}$ . The second corollary of Theorem 1 is essentially in D. J. Hartfiel [2].

COROLLARY 2. Let  $M = (\mu_{ij})$  be an  $n$ -square matrix with  $\mu_{ij} \geq 0$  and  $X = \{(i, j); \mu_{ij} > 0\}$ . Then there exists one diagonal matrix  $D_b = (e^{b_1}, e^{b_2}, \dots, e^{b_n})$  with positive diagonal elements such that  $D_b M D_b^{-1}$  is in  $\mathcal{M}_s$ , if and only if  $X$  is a symmetric pattern. Furthermore, the set of  $b \in \mathbf{R}^E$  such that  $D_b M D_b^{-1}$  is in  $\mathcal{M}_s$  is exactly a translate of  $\text{Ker}[d \circ \pi]$  if nonempty.

PROOF. Easy, taking  $H = \pi \circ d[\mathbf{R}^E]$ .

If we want now to complete the Corollary 2, we have to characterise  $\text{Ker}[\pi \circ c]$  for a given  $X \subset E \times E$  and characterise symmetric patterns.

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<sup>1</sup> NOTE ADDED IN PROOF. Professor Caussinus from the University of Toulouse has indicated to the author that a complete story of Corollary 1 can be found in Chapter 4 of M. Bacharach's monograph, *Biproportional matrices and input-output change*, Cambridge University Press, 1970. The first proof of Corollary 1 must be credited to W. M. Gorman (1963).

We consider the oriented graph with  $E$  as the set of vertices and  $X$  as the set of arcs.

PROPOSITION 2.  *$X$  is a symmetric pattern if and only if for any  $e \in E$  there exists a sequence  $e_0, e_1, \dots, e_n$  in  $E$  such that  $e_0=e_n=e$  and  $(e_{i-1}, e_i) \in X$  for  $i=1, \dots, n-1$  ( $n \geq 1$ ) (in terms of the graph  $(E, X)$  every vertex belongs to a circuit).*

Now we suppose that  $X$  is a symmetric pattern and we partition the set of vertices of the graph  $(E, X)$  into connected components  $(E_\alpha)_{\alpha \in A}$  where  $A$  is some index set and  $E_\alpha \neq \emptyset$ . These components correspond to “irreducible matrices” of [2]. We say that  $E_\alpha$  is the partition of  $E$  associated to the symmetric pattern  $X$ .

PROPOSITION 3. *Let  $X$  be a symmetric pattern of  $E \times E$ ,  $(E_\alpha)_{\alpha \in A}$  the associated partition of  $E$ . Then  $\dim[\text{Ker}(\pi \circ d)] = \text{number of elements of } A$ . Furthermore  $b \in \text{Ker}(\pi \circ d)$  and  $i_1$  and  $i_2 \in E_\alpha$  implies  $b_{i_1} = b_{i_2}$ .*

PROOF. Similar to Proposition 1.

4. **Other applications.** Following the same vein, we may now consider the result of R. Sinkhorn [8], stating that if  $M$  is an  $(n, n)$  symmetric matrix with strictly positive coefficients there exists one diagonal matrix  $D$  with positive diagonal coefficients such that  $DMD \in \mathcal{M}(r, r)$ , where  $r = (r_i)_{i=1}^n$  is given (Marcus and Newman [4] consider the doubly stochastic case).

Denote  $E = \{1, \dots, n\}$  and define the linear map  $e$  from  $\mathbf{R}^E$  to  $\mathbf{R}^{E \times E}$  by:

$$e[(b_i)_{i \in E}] = (b_i + b_j)_{(i, j) \in E \times E}.$$

If  $X$  is a subset of  $E \times E$ , call  $X$  an  $r$ -pattern if there exists a symmetric matrix  $(a_{ij})$  of  $\mathcal{M}(r, r)$  such that  $X = \{(i, j); a_{ij} > 0\}$ .

COROLLARY 3. *Let  $M = (\mu_{ij})$  be a symmetric matrix with  $\mu_{ij} \geq 0$ ,  $r = (r_i)_{i=1}^n$  a sequence such that  $r_i \geq 0$  and  $X = \{(i, j); \mu_{ij} > 0\}$ . Then there exists one diagonal matrix  $D_b = (e^{b_1}, \dots, e^{b_n})$  with positive diagonal elements such that  $D_b M D_b$  is in  $\mathcal{M}(r, r)$  if and only if  $X$  is an  $r$ -pattern. Furthermore, the set of  $b \in \mathbf{R}^E$  such that  $D_b M D_b$  is in  $\mathcal{M}(r, r)$  is exactly a translate of  $\text{Ker}[e \circ \pi]$  if nonempty.*

PROOF. Easy, taking  $H = r \circ e[\mathbf{R}^E]$ .

A similar study of dimension of  $\text{Ker}[\pi \circ e]$  can be done considering the number of components of the graph  $(E, X)$ ; characterisation of  $r$ -patterns depends on  $r$  in an unknown way, except in the doubly stochastic case, which is easy to work.

A trivial application of Theorem 1 is conditional expectation on a finite probability space: Let  $(X_\alpha)_{\alpha \in A}$  be a partition of  $X$ ; consider the boolean algebra  $\mathfrak{A}$  generated by  $(X_\alpha)_{\alpha \in A}$  and take  $H$  as space of  $\mathfrak{A}$ -measurable functions.

5. **Proof of Theorem 1.** Consider the function  $F$  defined on  $H$  by:

$$F(h) = \sum_{x \in X} \mu(x) \exp h(x).$$

Since  $\mu(x) > 0$  for all  $x$ ,  $F$  is strictly convex. One can easily check that  $F$  is continuously differentiable and that  $\text{grad } F_h$ , element of the dual space  $H^*$  of  $H$ , is given by:

$$\text{grad } F_h(g) = \sum_{x \in X} \mu(x) g(x) \exp h(x)$$

for all  $g$  in  $H$ . Now to each  $f$  of  $\mathbf{R}^X$ , we associate one point  $h_f^*$  of  $H^*$  defined by:

$$h_f^*(g) = \sum_{x \in X} \mu(x) g(x) \exp f(x) \quad \text{for all } g \in H.$$

We have to show that there exists one and only one point  $h_f \in H$  such that  $h_f^* = \text{grad } F_{h_f}$ .

Now we introduce the conjugate function  $F^*$  of  $F$ , valued in  $\mathbf{R} \cup \{+\infty\}$  and defined on  $H^*$  by:

$$F^*(h^*) = \sup_{h \in H} [h^*(h) - F(h)].$$

We denote by  $C^*$  the interior of  $\{h^*; F^*(h^*) < \infty\}$ . We use the following inequality:

$$be^a - e^b \leq a(e^a - 1) \quad (a, b \in \mathbf{R})$$

(to see this, replace  $b$  with  $a+b$ ) to get  $F^*(h_f^*) \leq \sum_{x \in X} \mu(x) f(x) (e^{f(x)} - 1)$  for all  $f$  in  $\mathbf{R}^X$ . Clearly the range of  $f \mapsto h_f^*$  in  $H^*$  is an open set, so  $h_f^* \in C^*$ . We made an appeal to Rockafellar [6, p. 258, Theorem 26.5]: we have shown that  $(H, F)$  and  $(C^*, F^*)$  are convex functions of Legendre type; from the quoted theorem we can assert that the map  $h \mapsto \text{grad } F_h$  from  $H$  to  $H^*$  is injective and that the range of this map is  $C^*$ ; this ends the proof.

6. **Other results and remarks.** The trivial example of conditional expectations quoted at the end of §4 leads us to ask if  $P_H: \mathbf{R}^X \rightarrow H$  defined by  $P_H f = h_f$  by means of Theorem 1 has some properties of ordinary linear projection.

**THEOREM 2.** *If  $P_H: \mathbf{R}^X \rightarrow H$  is defined by  $P_H f = h_f$  and if  $H$  and  $H'$  are two linear subspaces of  $\mathbf{R}^X$ , such that  $H \subset H'$ , then  $P_H = P_H P_{H'}$ .*

PROOF. Define  $G_H: R^X \rightarrow H^*$  by  $(G_H f)(g) = \sum_{x \in X} e^{f(x)} g(x) \mu(x)$  and  $\text{grad } F_H: H \rightarrow H^*$  as in the proof of Theorem 1. We know that  $\text{grad } F_H \circ G_H = G_H$  and we have to prove

$$\text{grad } F_H \circ P_H \circ P_{H'} = G_H, \text{ or } G_H \circ P_{H'} = G_H,$$

which is trivially true when  $H \subset H'$ .

From now,  $H$  is fixed and we denote  $\sum_{x \in X} f(x) \mu(x)$  by  $\int f d\mu$ . Notations are those of Theorem 1.

THEOREM 3. *If  $H$  contains constant functions of  $X$ , then:*

$$\int e^f d\mu \geq \int e^{h_f} h_f d\mu$$

and the equality holds only if  $f$  is in  $H$ .

PROOF. From the inequality  $t \geq 1 - e^{-t}$ , for  $t$  in  $R$  (strict if  $t \neq 0$ ), we get

$$\int e^f (1 - e^{h_f - f}) d\mu \leq \int e^f (f - h_f) d\mu.$$

Since  $H$  contains constants, the first member of the inequality is 0 (let  $g=1$  in the statement of Theorem 1). Now  $\int e^f h_f d\mu = \int e^{h_f} h_f d\mu$  (let  $g=h_f$ ), and we are done. Case of equality is clear.

THEOREM 4. *Let  $h$  in  $H$  and  $T: X \rightarrow X$ . Suppose that either  $h \circ T$  is in  $H$  or  $H$  contains constants and  $\mu \circ T = \mu$ . Then  $h_{h \circ T} = h$  implies  $h = h \circ T$ .*

PROOF. If  $h \circ T$  is in  $H$ , we use uniqueness of  $h_f$  in Theorem 1. In the other case, we apply Theorem 2 to  $f = h \circ T$ . But, since  $\mu \circ T = \mu$ , we have

$$\int e^f d\mu = \int e^{h_f} d\mu;$$

this is the case of equality of Theorem 3.

REMARKS. Choose  $X = \{1, \dots, n\} \times \{1, \dots, m\}$ ,  $\mu(x) \equiv 1$  and  $H$  as in §2. If  $f$  in  $R^X$  is such that  $\int e^f d\mu = 1$ , Theorem 3 gives an inequality about entropy of the joint law of probability, well known in information theory (see [5, pp. 146–157]). If  $T$  is a permutation of  $X$ , of course  $\mu \circ T = \mu$ ; a nice interpretation of the second part of Theorem 4 is as follows.

Let  $r = (\exp b_i)_{i=1}^n$  and  $s = (\exp b'_j)_{j=1}^m$  such that

$$\sum_{i=1}^n \exp b_i = \sum_{j=1}^m \exp b'_j = 1.$$

Put the  $nm$  numbers  $\exp(b_i + b'_j)$  on slips of paper, try to put them in an  $(n, m)$  matrix in order to get an element of  $\mathcal{M}(r, s)$ ; you always get the matrix  $(\exp(b_i + b'_j))$ , that is to say, the product distribution of probability. Of course we need only Theorem 3, not Theorem 1, to get this result, which can be generalised to a product of countable spaces, with distributions  $r$  and  $s$  such that  $\sum_i e^{b_i} b_i < \infty$ ,  $\sum_j e^{b'_j} b'_j < \infty$ . I think this is not true if  $\sum_i e^{b_i} b_i = \infty$ , but I know of no counterexamples.

Considering the exponential in Theorem 1 suffices for applications, but we could replace the exponential by any convex function  $\varphi$  positive and increasing, with primitive  $\psi$ . The proof of Theorem 1 would start from  $F(h) = \sum_x \psi(h(x))\mu(x)$ .

Generalisations of Theorem 1 to measured spaces  $(X, \mu)$  will be done in a forthcoming paper.

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