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ON LOCAL SOLVABILITY OF PSEUDO-DIFFERENTIAL EQUATIONS

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ABSTRACT. A sufficient condition for the local solvability of the equation $u_t - A(x, t, D_x)u = f(x, t)$ is proved, where A is a first order pseudo-differential operator with real symbol. This is a special case of the local solvability conjecture of Nirenberg and Treves.

Introduction. Let $P(x, D)$ be a linear partial differential operator of principal type with smooth complex value coefficients. The question of when the equation $Pu=f$ is locally solvable has been settled by Nirenberg and Treves [4] and Beals and Fefferman [1]. Local solvability is equivalent to the condition:

(P) The imaginary part of P does not change signs on the null bicharacteristics of the real part of P.

For a pseudo-differential operator Nirenberg and Treves conjectured that local solvability is equivalent to the condition:

(Ψ) On every null bicharacteristic of $\operatorname{Re} P$, if $\operatorname{Im} P$ is negative at a point it remains nonpositive from then on.

The purpose of this note is to prove the following special case of this conjecture.

THEOREM 1. *Let $P = d/dt - A(t, x, D_x)$ for $(t, x) \in \Omega$ where A is a first order pseudo-differential operator with real symbol $a(t, x, \xi)$. Assume that (Ψ) if $a(t_0, x_0, \xi_0) < 0$ then for $t > t_0$, $a(t, x_0, \xi_0) \leqq 0$; and if $a(t_0, x_0, \xi_0) = 0$ then* $\operatorname{grad}_{x,\xi} a(t_0, x_0, \xi_0) = 0$.

Then P is locally solvable.

Theorem 1 is a simple consequence of the following a priori estimate for the adjoint of P.

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THEOREM 2. *Let* $L=d/dt-a(t, x, D_x)$ *for* $(t, x) \in \Omega = I \times U \subset R^{n+1}$ *(I is an interval containing* $t=0$*) where A is a first order pseudo-differential operator with real symbol* $a(t, x, \xi)$*. Assume that if* $a(t_0, x_0, \xi_0)>0$ *then* $a(t, x_0, \xi_0) \geqq 0$ *for* $t>t_0$*. Also assume that if* $a=0$ *that* $\operatorname{grad}_{x,\xi} a=0$*. Then given* $\varepsilon>0$*, there exists a* δ *such that*

$$\|u\|_0 \leqq \varepsilon \|Lu\|_0 \quad \textit{for all } u \in C_0^\infty((-\delta, \delta) \times U).$$

The proof of Theorem 2 follows the lines of the constant coefficient case (cf. Nirenberg-Treves [5]). To do this first we localize A as in Hörmander [2]. We need to prove an estimate similar to the sharp Gårding's inequality, but when the symbol does not have constant sign. Lax-Nirenberg [3] have observed that the positivity of the symbol is needed in the proof of Gårding's inequality only to establish that $|\operatorname{grad}_\xi a(x, \xi)|^2 \leqq C|a(x, \xi)| \, |\xi|^{-1}$. But this is a consequence of our second assumption about a.

Proof of Theorem 2. We use the notation of Hörmander [2]. In particular $a_\beta^\alpha=(iD_x)^\beta (iD_\xi)^\alpha a(x, \xi)$.

LEMMA 1. *If* $a(x, \xi)=0$ *implies* $\operatorname{grad}_{x,\xi} a(x, \xi)=0$ *then*

$$|\operatorname{grad}_\xi a(x, \xi)|^2 (1+|\xi|) + |\operatorname{grad}_x a(x, \xi)|^2 (1+|\xi|)^{-1} \leqq C\, |a(x, \xi)|.$$

PROOF. If $f(x)=0$ implies $f'(x)=0$ then $|f(x)|$ is a nonnegative C^2 function. Hence $|f'(x)|^2 \leqq 2|f(x)| \max |f''|$. Applying this inequality to each variable in question and using the pseudo-homogeneity of a we get the lemma.

Hereafter we make the convention that C is any constant depending only on the symbol of A.

We now introduce Hörmander's partition of unity. Construct nonnegative functions $\phi_j(x) \in C_0^\infty(R^n)$ such that $\sum_{j=1}^\infty \phi_j^2(x)=1$ and $x, y \in \operatorname{supp} \phi_j$ implies that $|x-y| \leqq C$, and the supports overlap a bounded number of times. Set $\psi_j(\xi)=\phi_j(\xi|\xi|^{-1/2})$ also in $C_0^\infty(R^n)$. The important properties of the ψ_j are that $\sum \psi_j^2=1$,

$$\xi, \eta \in \operatorname{supp} \psi_j \text{ implies } |\xi-\eta| < C\, |\xi|^{1/2} \quad \text{and} \tag{2}$$

$$\sum_{j=1}^\infty |\psi_j(\eta)-\psi_j(\xi)|^2 \leqq \frac{C\,|\xi-\eta|^2}{|\xi|} + \frac{8\,|\xi-\eta|}{|\xi|}. \tag{3}$$

(See Hörmander [2, pp. 141–142] for proofs.) Let ξ_j be any point in the support of ψ_j and set $u^{jk}(x)=\phi_k(x|\xi_j|^{1/2})\psi_j(D)u$ and $\phi_{jk}=\psi_k(x|\xi_j|^{1/2})$. Observe that

$$\sum_{j,k} \|u^{jk}\|_0^2 = \|u\|_0^2. \tag{4}$$

Choose x^{jk} an arbitrary point in the support of ϕ_{jk}.

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LEMMA 2. *Depending on whether* $a(x^{jk}, \xi^j) \geqq 0$ *or* <0,

$$\operatorname{Re}(\psi^{jk}(x)\psi_j(D)Au, \psi^{jk}(x)\psi_j(D)u) \geqq R_{jk}(t) \quad \text{or} \quad \leqq R_{jk}(t)$$

where $\sum_{j,k} |R_{jk}(t)| \leqq C\|u(t)\|_0^2$.

PROOF OF THEOREM 2. Let $Lu=f$. By the first assumption on a, for each j, k there is a t_{jk} such that $a(t, x_{jk}, \xi) \leqq 0$ for $t<t_{jk}$ and $\geqq 0$ for $t>t_{jk}$. Let $u \in C_0^\infty((-\delta, \delta)\times U)$, δ to be determined in a moment. If $a(t, x_{jk}, \xi_j) \leqq 0$ we have that,

$$\begin{aligned} \frac{d\,\|u^{jk}\|^2}{dt} &= 2\operatorname{Re}(\phi^{jk}(x)\psi_j(D)u_t, u^{jk}) \\ &= 2\operatorname{Re}(\phi^{jk}\psi_j(D)(Au+f), u^{jk}) \\ &\leqq 2R_{jk} + \|\phi^{jk}(x)\psi_j(D)f\|^2 + \|u_{jk}\|^2 \end{aligned}$$

Upon integrating the above inequality from $-\delta$ to t_{jk} we get

$$\|u^{jk}(t)\|^2 \leqq \int_{-\delta}^{t_{jk}} 2R_{jk}(t) + \|\phi^{jk}\psi_j f\|^2 + \|u^{jk}\|^2\,dt.$$

Similarly when $a(x^{jk}, \xi_j) \geqq 0$ we get

$$\|u^{jk}(t)\|^2 \leqq \int_{t_{jk}}^{\delta} 2R_{jk}(t) + \|\phi^{jk}\psi_j f\|^2 + \|u^{jk}\|^2\,dt.$$

Combining the last two inequalities, summing over j and k and applying Lemma 2 and equation (4) yields that

$$\begin{aligned} \|u\|_0^2 = \sum_{jk} \|u^{jk}\|^2 &\leqq \int_{-\delta}^{\delta} \sum_{jk} R_{jk} + \|\phi^{jk}\psi_j(D)f\|^2 + \|u^{jk}\|^2\,dt \\ &\leqq \int_{-\delta}^{\delta} C\,\|u\|_0^2 + \|f(t)\|^2\,dt. \end{aligned}$$

If we choose δ small enough so that $4\delta C<1$ it follows that

$$\|u(t)\|^2 \leqq 2\int_{-\delta}^{\delta} \|f(t)\|^2\,dt$$

and integrating once more that

$$\int \|u(t)\|^2\,dt \leqq 2\delta \int_{-\delta}^{\delta} \|f(t)\|^2\,dt,$$

which proves Theorem 1.

PROOF OF LEMMA 2. First we will show that,

$$\sum_j |(\psi_j(D)Au, \psi_j(D)u) - (A\psi_j(D)u, \psi_j(D)u)| \leqq C\,\|u\|^2. \tag{6}$$

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Let $\hat{a}(\eta, \xi)$ be the Fourier transform of $a(x, \xi)$ with respect to x. We may assume that, outside of U, a has compact support in x, so it then follows that $|\hat{a}(\eta, \xi)| \leqq C_N|\xi|(1+|\eta|)^{-N}$ for any integer N. Using Parseval's theorem, the left-hand side of (6) equals

$$\iint \hat{a}(\eta-\xi, \xi)(\phi_j^2(\eta) - \psi_j(\eta)\psi_j(\xi))\hat{u}(\xi)\hat{u}(-\eta)\, d\xi\, d\eta$$
$$= \frac{1}{2}\iint \hat{a}(\eta-\xi, \xi)\,|\psi_j(\eta) - \psi_j(\xi)|^2\,\hat{u}(\xi)\hat{u}(-\eta)\, d\xi\, d\eta.$$

Taking absolute values, summing up over j and applying inequality (3) we may bound the left-hand side of (6) by

$$\iint \frac{C_N|\xi|}{(1+|\eta-\xi|)^N}\left[\frac{C|\eta-\xi|^2}{|\xi|} + 8\frac{|\xi-\eta|}{|\xi|}\right]|\hat{u}(\xi)|\,|\hat{u}(\eta)|\, d\eta\, d\xi \leqq C\,\|u\|_0^2$$
$$\text{if } N \geqq n+1.$$

Next setting $A_j = a(x, \xi_j) + \sum_{\nu=1}^n a^\nu(x, \xi_j)(D_\nu - \xi_j)$, we have that

$$|((A - A_j)\psi_j(D)u, \psi_j(D)u)| \leqq C\,\|\psi_j(D)u\|_0^2. \tag{7}$$

Parseval's theorem tells us that the left side of (7) equals

$$\iint\left\{\hat{a}(\eta-\xi, \xi) - \hat{a}(\eta-\xi, \xi_j) - \sum_\nu \hat{a}^\nu(\eta-\xi, \xi_j)(\xi_\nu - \xi_\nu^j)\right\}$$
$$\cdot\,\psi_j(\eta)\psi_j(\xi)u(-\eta)u(\xi)\, d\eta\, d\xi.$$

But $|\xi-\xi_j| \leqq C|\xi_j|^{1/2}$ when $\psi_j(\xi) \neq 0$.

Thus by Taylor's theorem the expression in parentheses is

$$O\left(\frac{|\xi_j|^{-1}}{1+|\eta-\xi|^N}|\xi-\xi_j|^2\right) = O((1+|\eta-\xi|)^{-N})$$

and the bound follows. Note that

$$\begin{aligned}\operatorname{Re}(\phi(x)A_ju, \phi u) &= \operatorname{Re}\left\{(A_j\phi u, \phi u) + i\int\sum_\nu a^\nu\phi(iD_\nu)\phi\,|u|^2\right\}\\ &= \operatorname{Re}(A_j\phi u, \phi u) \quad \text{since } a \text{ is real.}\end{aligned} \tag{8}$$

Combining (6), (7) and (8) we have that

$$\sum_{j,k}|\operatorname{Re}(\phi_{jk}(x)\psi_j(D)Au, u^{jk}) - \operatorname{Re}(A_ju^{jk}, u^{jk})| \leqq C\,\|u\|_0^2. \tag{9}$$

Let

$$u^{jk}(x) = \exp\{i\langle x, \xi_j\rangle\}v^{jk}((x-x^{jk})\,|\xi_j|^{1/2})$$

and

$$\psi_j(D)u = \exp\{i\langle x, \xi_j\rangle\}v^j(x\,|\xi_j|^{1/2}).$$

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Then $\phi_k(x)v^j$ and v^{jk} differ only by a translation; $|y| \leqq C$ if $y \in \operatorname{supp} v^{jk}$ and $|\xi| \leqq C$ if $\xi \in \operatorname{supp} v^j$. Therefore it follows that

$$\begin{aligned}\int \sum_{k;|\alpha+\beta|\leqq N} |y^\beta D^\alpha v^{jk}|^2\,dy &\leqq C \sum_{k;|\alpha|\leqq N} \int |D^\alpha v^{jk}|^2\,dy \\ &\leqq C \sum_{k;|\alpha|\leqq N} \int |D^\alpha \phi_k(x) v^j(y)|\,dy \\ &\leqq C \sum_{|\alpha|\leqq N} \int |D^\alpha v^j(y)|\,dy \\ &\leqq C_N\,|\xi_j|^{n/2}\,\|\psi_i(D)u\|_0^2.\end{aligned} \tag{10}$$

By a change of variables we see that

$$\begin{aligned}&(A_j u^{jk}, u^{jk}) \\ &\quad = |\xi_j|^{-n/2} \int v^{jk}(y) \Big\{ a(x^{jk} + y\,|\xi_j|^{-1/2}, \xi_j) \\ &\qquad\qquad + \sum_\nu a^\nu(x^{jk} + y\,|\xi_j|^{-1/2}, \xi_j)\,|\xi_j|^{1/2}\,D_\nu \Big\} v^{jk}(y)\,dy.\end{aligned}$$

By Taylor's theorem, the fact that supp v^{jk} is bounded, and (10), we have that

$$\begin{aligned}&\sum_k (A_j u^{jk}, u^{jk}) - |\xi_j|^{-n/2} \int v^{jk}(y) \Big\{ a(x^{jk}, \xi_j) + \sum_\nu a_\nu(x^{jk}, \xi_j)\,|\xi_j|^{-1/2}\,y \\ &\qquad\qquad + \sum_\nu a^\nu(x^{jk}, \xi_j)\,|\xi_j|^{1/2}\,D_\nu \Big\} v^{jk}(y)\,dy \\ &\qquad \leqq |\xi_j|^{-n/2} \sum_{k,|\beta|\leqq 2;|\alpha|\leqq 1} \int |y^\beta D^\alpha v^{jk}|\,dy \leqq C\,\|\psi_j(D)u\|_0^2.\end{aligned} \tag{11}$$

Finally we have, by using Lemma 1 and the Cauchy-Schwartz inequality and the bound on the support of v^{jk},

$$\begin{aligned}&\sum_\nu \int a^\nu(x^{jk}, \xi^j)\,|\xi^j|^{1/2}\,(D_\nu v^{jk}(y)) v^{ik}(y)\,dy \\ &\qquad = \sum_\nu \int \hat{a}^\nu(x^{jk}, \xi^j)\,|\xi^j|^{1/2}\,\xi_\nu\,|\hat{v}^{ik}(\xi)|^2\,d\xi \\ &\qquad \leqq \frac{1}{4} \int |a(x^{jk}, \xi^j)|\,|\hat{v}^{jk}(\xi)|^2\,d\xi + \int \frac{C}{|\xi_j|}\,|\xi^j|\,|\xi|^2\,|\hat{v}^{jk}(\xi)|^2\,d\xi \\ &\qquad \leqq \frac{1}{4} \int |a(x^{jk}, \xi_j)|\,|v(y)|^2\,dy + |\xi_j|^{n/2}\,\|u^{jk}\|_0^2.\end{aligned}$$

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Similarly,

$$\sum_{\nu} a_\nu(x^{jk}, \xi^j)\, |\xi_j|^{-1/2}\, y\, |v^{jk}(y)|^2\, dy$$
$$\leqq \frac{1}{4}\int |a(x^{jk}, \xi^j)|\, |v^{jk}(y)|^2 + \int Cy^2\, |v^{jk}(y)|^2\, dy$$
$$\leqq \frac{1}{4}\, |a(x^{jk}, \xi^j)|\, |v^{jk}(y)|^2\, dy + |\xi_j|^{n/2}\, \|u^{jk}\|_0^2.$$

If $a(x^{jk}, \xi^j) \geqq 0$ we may combine this with (9) and (11) to get

$$(\phi^{jk}(x)\psi_j(D)Au, u^{jk}) \geqq -R_{jk}$$

and $\sum |R_{jk}| \leqq C|u|_0^2$. This and similar considerations for the case $a(x^{jk}, \xi^j) <$ 0 completes the proof of the lemma.

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